Brauer–Manin Obstruction for Indefinite Integral Quadratic Forms Representing Integers

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Abstract

Classical Hasse principle guarantees that a smooth quadratic hypersurface of dimension at least one over a number field has a rational point if and only if it has points over all completions. For integral points, however, local solubility everywhere does not always imply global solubility. This report, following Colliot-Thélène and Xu, explains how such failures can be systematically described by an *integral Brauer–Manin obstruction*. We reformulate the problem of representing an integer a by an indefinite integral quadratic form $f(x_1, \ldots, x_m)$ as the existence of integral points on a certain $\mathbb{Z}$-scheme $\mathbf{X}$. The generic fiber $X = \mathbf{X} \times_{\mathbb{Z}} \mathbb{Q}$ is a homogeneous space G/H with $G = \mathrm{Spin}(f)$. We compute the Brauer group of X, describe explicit Azumaya algebras giving the obstruction, and prove the main theorem: when f is indefinite and $m \geq 3$, global integral points exist if and only if the Brauer–Manin set is nonempty. Two detailed examples are worked out: one where the obstruction vanishes (no integral solution), and one where the form is positive definite, showing that the indefiniteness hypothesis is essential.

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1 Introduction

The representation of an integer by an integral quadratic form is one of the oldest problems in number theory. Given a nondegenerate integral quadratic form f of rank m and a nonzero integer a, we ask whether there exist integers $x_1, \ldots, x_m$ such that

$$f(x_1, \ldots, x_m) = a.$$

When $m \geq 3$, the Hasse–Minkowski theorem asserts that the corresponding equation has a *rational* solution over $\mathbb{Q}$ if and only if it has solutions in all completions $\mathbb{Q}_v$ (including $\mathbb{R}$). However, the analogous statement for *integral* solutions is false in general. Counterexamples are abundant in the literature, and they are often explained through concepts such as spinor exceptions, genera, and spinor genera.

In their remarkable paper [2], Colliot-Thélène and Xu provided a unified geometric framework for these phenomena. They introduced an *integral Brauer–Manin obstruction* for the existence of integral points on schemes defined over the ring of integers of a number field. Applied to the representation problem, this obstruction turns out to be the only obstruction when the quadratic form is indefinite.

We keep the exposition over $\mathbb{Z}$ and $\mathbb{Q}$ for simplicity; the original paper treats the general number field case. The report is organized as follows. Section 2 sets up the notation and recalls basic facts about integral quadratic forms, schemes, Brauer groups, and the Brauer–Manin pairing. Section 3 studies the homogeneous space structure of the generic fiber and computes its Brauer group. Section 4 states the main theorem and sketches its proof. Sections 5 and 6 provide two illustrative examples: one indefinite case where the obstruction is the only reason for non-existence of integral solutions, and one definite case where the obstruction vanishes but no integral point exists, highlighting the necessity of the indefiniteness hypothesis.

2 Preliminaries

2.1 Indefinite integral quadratic forms

Definition 2.1. *Let $m \geq 1$. An* integral quadratic form *is a homogeneous polynomial of degree two*

$$f(x_1, \ldots, x_m) = \sum_{i,j=1}^{m} a_{ij} x_i x_j, \qquad a_{ij} = a_{ji} \in \mathbb{Z},$$

which is non-degenerate, i.e. the symmetric matrix (a_{ij}) has nonzero determinant. By regarding the coefficients as real numbers, we obtain a real quadratic form; let (p, q) be its signature, where p (resp. q) is the number of positive (resp. negative) eigenvalues, $p + q = m$. If $p > 0$ and $q > 0$, then f is called indefinite. *In other words, f is neither positive definite nor negative definite over $\mathbb{R}$.*

2.2 Schemes, integral points and generic fibre

Let $O = \mathbb{Z}$, $k = \mathbb{Q}$. For a given $a \in \mathbb{Z} \setminus \{0\}$, we define the affine O-scheme

$$\mathbf{X} = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_m]/(f(x_1, \ldots, x_m) - a).$$

For any $\mathbb{Z}$-algebra A, the set $\mathbf{X}(A) = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Spec} A, \mathbf{X})$ is the set of A-points. In particular:

- $\mathbf{X}(\mathbb{Z})$ corresponds to integer solutions of $f(\mathbf{x}) = a$;
- $\mathbf{X}(\mathbb{Z}_p)$ corresponds to p-adic integer solutions;
- $\mathbf{X}(\mathbb{R})$ corresponds to real solutions.

The generic fibre $X = \mathbf{X} \times_{\mathbb{Z}} \mathbb{Q}$ is a smooth affine $\mathbb{Q}$-variety whose $\mathbb{Q}$-points are rational solutions of the equation. Since $m \geq 3$ and local (real and p-adic) solutions exist everywhere, the classical Hasse principle implies $X(\mathbb{Q}) \neq \varnothing$, so a rational solution always exists under our assumptions.

2.3 Brauer groups

For a field F, the Brauer group $\operatorname{Br} F$ is the group of similarity classes of finite-dimensional central simple algebras over F, with tensor product as the group operation. For a scheme Y, $\operatorname{Br} Y = H^2_{\text{ét}}(Y, \mathbb{G}_m)$ (étale cohomology). When $Y = \operatorname{Spec} F$ these two definitions agree.

For a local field k_v ($\mathbb{R}$ or $\mathbb{Q}_p$), local class field theory provides a canonical embedding, the *local invariant*,

$$\operatorname{inv}_v \colon \operatorname{Br} k_v \hookrightarrow \mathbb{Q}/\mathbb{Z},$$

which is an isomorphism for p-adic fields and has image $\{0, \frac{1}{2}\}$ for $\mathbb{R}$. The global reciprocity law is the exact sequence

$$0 \to \operatorname{Br} \mathbb{Q} \to \bigoplus_v \operatorname{Br} \mathbb{Q}_v \xrightarrow{\sum \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0,$$

where the sum runs over all places v of $\mathbb{Q}$.

2.4 Integral adeles and the Brauer–Manin pairing

Let S be a finite set of places containing ∞ and all primes where $\mathbf{X}$ fails to be smooth or the discriminant of f is not a unit. The *integral adele space* is

$$\mathbf{X}(\mathbb{Q}) = \prod_{v \in S} X(\mathbb{Q}_v) \times \prod_{p \notin S} \mathbf{X}(\mathbb{Z}_p).$$

The *Brauer–Manin pairing* is

$$\langle \cdot, \cdot \rangle : \mathbf{X}(\mathbb{Q}) \times \operatorname{Br} X \to \mathbb{Q}/\mathbb{Z}, \qquad (\{P_v\}, \alpha) \mapsto \sum_v \operatorname{inv}_v(\alpha(P_v)),$$

where $\alpha(P_v)$ is obtained by pulling back α via $P_v : \operatorname{Spec} k_v \to X$. Class field theory implies that the diagonal image of $\mathbf{X}(\mathbb{Z})$ is contained in the left kernel of this pairing. Define the *Brauer–Manin set*

$$\mathbf{X}(\mathbb{Q})^{\operatorname{Br} X} = \big\{ \{P_v\} \in \mathbf{X}(\mathbb{Q}) \mid \langle \{P_v\}, \alpha \rangle = 0 \ \forall\, \alpha \in \operatorname{Br} X \big\}.$$

If $\mathbf{X}(\mathbb{Q}) \neq \varnothing$ (local solutions everywhere) but $\mathbf{X}(\mathbb{Q})^{\operatorname{Br} X} = \varnothing$, we say there is an *integral Brauer–Manin obstruction* to the existence of an integral point.

3 Geometry of the equation as a homogeneous space

3.1 Homogeneous space structure

Let $V = \mathbb{Q}^m$ be equipped with the quadratic form $q(v) = f(v)$. The associated symmetric bilinear form is $B(u, v) = \frac{1}{2}(q(u + v) - q(u) - q(v))$. Fix a rational point $P_0 \in X(\mathbb{Q})$ corresponding to a vector $v_0 \in V$ with $q(v_0) = a$.

The special orthogonal group $\mathrm{SO}(q)$ acts transitively on X by Witt's extension theorem; the stabilizer of P_0 is $\mathrm{SO}(h)$, where h is the restriction of q to the orthogonal complement $v_0^{\perp}$ (of dimension $m - 1$). Thus

$$X \cong \mathrm{SO}(q)/\mathrm{SO}(h).$$

We pass to the spin double cover. There is a short exact sequence of algebraic groups

$$1 \to \mu_2 \to \mathrm{Spin}(q) \xrightarrow{\pi} \mathrm{SO}(q) \to 1,$$

where $\mathrm{Spin}(q)$ is semisimple and simply connected. Let $G = \mathrm{Spin}(q)$ act on X via π; the action remains transitive. The stabilizer H fits into

$$1 \to \mu_2 \to H \to \mathrm{SO}(h) \to 1.$$

Hence

$$X \cong G/H, \qquad G = \mathrm{Spin}(q).$$

The structure of H depends on $m - 1$ (the rank of h):

- If $m \geq 4$ ($m - 1 \geq 3$), then $H = \mathrm{Spin}(h)$ is semisimple and simply connected.
- If $m = 3$ ($m - 1 = 2$), then $\mathrm{SO}(h)$ is a one-dimensional torus, hence H is a one-dimensional torus T. Let $d = -a \cdot \mathrm{disc}(q)$ and $K = \mathbb{Q}(\sqrt{d})$. When d is not a square in $\mathbb{Q}$, T is the double cover of the norm-1 torus $\mathrm{R}^1_{K/\mathbb{Q}}\mathbb{G}_m$.
- The case $m = 2$ is outside our main discussion.

From now on we focus on the case $m = 3$, which is the classical problem of representing an integer by a ternary quadratic form.

3.2 The Brauer group of X

We now compute $\mathrm{Br}\, X$ modulo the image of $\mathrm{Br}\, \mathbb{Q}$ using the homogeneous space description $X = G/H$ with $G = \mathrm{Spin}(q)$ semisimple simply connected and H connected (for $m \geq 3$). The key tool is a general exact sequence due to Sansuc.

Proposition 3.1 (Sansuc, [4, Proposition 6.10])**.** *Let H be a connected linear algebraic group over a field k of characteristic 0, and let $p : Y \to X$ be a torsor over a smooth connected k-variety X under H. There is a natural exact sequence of abelian groups*

$$0 \to k[X]^*/k^* \to k[Y]^*/k^* \to \widehat{H}(k) \to \mathrm{Pic}\, X \to \mathrm{Pic}\, Y \to \mathrm{Pic}\, H \to \mathrm{Br}\, X \to \mathrm{Br}\, Y,$$

where $\widehat{H}$ is the (geometric) character group of H (a finitely generated, $\mathbb{Z}$-free, discrete Galois module). The group $\mathrm{Pic}\, H$ *is finite.*

Apply this to our situation with $k = \mathbb{Q}$, $X = G/H$ as above, and $Y = G$ viewed as an H-torsor over X via the quotient map $G \to G/H$. Note that G is a smooth connected $\mathbb{Q}$-variety. Let us check the terms in the exact sequence.

Lemma 3.2. *For a semisimple simply connected group G over $\mathbb{Q}$, we have*

$$\mathbb{Q}[G]^* = \mathbb{Q}^* \quad (\textit{no nonconstant invertible regular functions}),$$

and over $\overline{\mathbb{Q}}$, $\overline{\mathbb{Q}}[G_{\overline{\mathbb{Q}}}]^* = \overline{\mathbb{Q}}^*$.

Proof. This is standard: a semisimple simply connected group has no nontrivial characters, and its ring of global functions consists only of constants (see [2] Proposition 2.6). □

Similarly, for the homogeneous space $X = G/H$, one can show that $\mathbb{Q}[X]^* = \mathbb{Q}^*$ as well (because G is semisimple and H is geometrically a stabilizer of a point in a G-variety with trivial units). Thus both $k[X]^*/k^*$ and $k[Y]^*/k^*$ vanish.

Lemma 3.3 (Properties of G)**.** *For $G = \mathrm{Spin}(q)$ semisimple simply connected over $\mathbb{Q}$:*

(i) $\operatorname{Pic} G = 0$*;*

(ii) the natural map $\operatorname{Br}\mathbb{Q} \to \operatorname{Br} G$ *is an isomorphism.*

Proof. (i) Follows from Proposition 2.6 of [2] because G is semisimple simply connected and $\operatorname{Pic}\overline{G} = 0$. (ii) By Proposition 2.6, $\operatorname{Br}\overline{G} = 0$, and the Hochschild–Serre spectral sequence together with $\overline{\mathbb{Q}}[G]^* = \overline{\mathbb{Q}}^*$ and $\operatorname{Pic}\overline{G} = 0$ gives $\operatorname{Br} G \cong \operatorname{Br}\mathbb{Q}$. More concretely, every element of $\operatorname{Br} G$ becomes trivial over $\overline{\mathbb{Q}}$, so it belongs to $\operatorname{Br}_1 G$, and the spectral sequence $E_2^{pq} = H^p(\mathbb{Q}, H^q_{\text{ét}}(\overline{G}, \mathbb{G}_m))$ collapses because $H^1(\overline{G}, \mathbb{G}_m) = \operatorname{Pic}\overline{G} = 0$ and $H^0(\overline{G}, \mathbb{G}_m) = \overline{\mathbb{Q}}^*$, leading to $\operatorname{Br} G/\operatorname{Br}\mathbb{Q} \hookrightarrow H^1(\mathbb{Q}, \operatorname{Pic}\overline{G}) = 0$, hence $\operatorname{Br} G = \operatorname{Br}\mathbb{Q}$. □

Plugging these vanishing results into Sansuc's sequence with $Y = G$ and $X = G/H$, we obtain the exact sequence

$$0 \to \widehat{H}(\mathbb{Q}) \to \operatorname{Pic} X \to 0 \to \operatorname{Pic} H \to \operatorname{Br} X \to \operatorname{Br} G = \operatorname{Br}\mathbb{Q} \to \dots$$

where the map $\operatorname{Pic} X \to \operatorname{Pic} G = 0$ forces $\operatorname{Pic} X$ to be a quotient of $\widehat{H}(\mathbb{Q})$. Actually, from the exactness, we get

$$\widehat{H}(\mathbb{Q}) \xrightarrow{\cong} \operatorname{Pic} X, \tag{2}$$

and then

$$0 \to \operatorname{Pic} H \to \operatorname{Br} X \to \operatorname{Br}\mathbb{Q} \to \cdots$$

Since the sequence is exact and $\operatorname{Br}\mathbb{Q} \to \operatorname{Br} X$ is injective (as $\operatorname{Br} X$ contains $\operatorname{Br}\mathbb{Q}$ via pullback), we obtain a short exact sequence

$$0 \to \operatorname{Pic} H \to \operatorname{Br} X/\operatorname{Br}\mathbb{Q} \to 0,$$

i.e. an isomorphism

$$\operatorname{Br} X/\operatorname{Br}\mathbb{Q} \cong \operatorname{Pic} H. \tag{3}$$

This is the "fundamental result of Sansuc" (formulated as Proposition 2.10 in [2]) for our specific homogeneous space.

It remains to compute $\operatorname{Pic} H$ in the cases of interest.

- If $m \geq 4$, then $H = \mathrm{Spin}(h)$ is semisimple simply connected of rank ≥ 3, so $\operatorname{Pic} H = 0$ by Lemma 3.3(i). Hence $\operatorname{Br} X / \operatorname{Br} \mathbb{Q} = 0$; there is no algebraic Brauer obstruction.

- If $m = 3$, then $H = T$ is a one-dimensional torus. For a torus, $\operatorname{Pic} T$ is canonically isomorphic to $H^1(\mathbb{Q}, \widehat{T})$, where $\widehat{T}$ is the Galois module of characters. Let $d = -a \cdot \mathrm{disc}(q)$; then $K = \mathbb{Q}(\sqrt{d})$ and T is the double cover of $\mathrm{R}^1_{K/\mathbb{Q}}\mathbb{G}_m$.

  - If d is a square in $\mathbb{Q}$, then T is split, so $\widehat{T} \cong \mathbb{Z}$ with trivial Galois action, hence $H^1(\mathbb{Q}, \widehat{T}) = 0$ and $\operatorname{Pic} T = 0$. Consequently $\operatorname{Br} X / \operatorname{Br} \mathbb{Q} = 0$.
  - If d is not a square, then $\widehat{T} \cong \mathbb{Z}[G]/\mathbb{Z}(1+\sigma)$ where $G = \mathrm{Gal}(K/\mathbb{Q}) = \{1, \sigma\} \cong \mathbb{Z}/2\mathbb{Z}$. A direct calculation gives $H^1(\mathbb{Q}, \widehat{T}) \cong \mathbb{Z}/2\mathbb{Z}$. Therefore $\operatorname{Pic} T \cong \mathbb{Z}/2\mathbb{Z}$, and by (3)
    $$\operatorname{Br} X / \operatorname{Br} \mathbb{Q} \cong \mathbb{Z}/2\mathbb{Z}.$$
    There exists (up to addition of an element from $\operatorname{Br} \mathbb{Q}$) a unique nontrivial algebraic Brauer class $\alpha \in \operatorname{Br} X$ of order 2 generating this quotient.

This completes the computation of the Brauer group.

3.3 Explicit generator for $m = 3$

Assume $m = 3$ and d is not a square. Fix a rational point $M = (x_0, y_0, z_0) \in X(\mathbb{Q})$. The tangent plane polynomial is

$$L(x, y, z) = B(M, (x, y, z)) - a.$$

The quaternion algebra $\mathcal{A} = (L, d)$ defined on the function field $\mathbb{Q}(X)$ extends to an Azumaya algebra on the whole of X; its class $\alpha \in \operatorname{Br} X$ generates $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$. For any local point $P \in X(\mathbb{Q}_v)$ with $L(P) \neq 0$,

$$\mathrm{inv}_v(\alpha(P)) = \big(L(P), d\big)_v \in \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z},$$

where $(\cdot, \cdot)_v$ is the Hilbert symbol on $\mathbb{Q}_v$.

4 The main theorem

We keep the notation of the previous sections: f is an indefinite integral quadratic form of rank $m \geq 3$, $a \in \mathbb{Z} \setminus \{0\}$, and $\mathbf{X} = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_m]/(f - a)$. We assume everywhere local integral solubility, i.e. $\mathbf{X}(_{\mathbb{Q}}) \neq \varnothing$.

Recall that $X = \mathbf{X} \times_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the homogeneous space G/H with $G = \mathrm{Spin}(f)$ semisimple simply connected and H the stabilizer of a chosen rational point. According to the rank we have:

- If $m \geq 4$, then $H = \mathrm{Spin}(h)$ is semisimple simply connected; $\operatorname{Pic} H = 0$ and $\operatorname{Br} X = \operatorname{Br} \mathbb{Q}$.

- If $m = 3$, then $H = T$ is a one-dimensional torus; $\operatorname{Br} X / \operatorname{Br} \mathbb{Q} \cong \operatorname{Pic} T$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$.

Theorem 4.1 (Integral Brauer–Manin obstruction, [2] Theorems 6.1, 6.3, 6.4)**.** *Under the above assumptions,*

$$\mathbf{X}(\mathbb{Z}) \neq \varnothing \iff \mathbf{X}(_{\mathbb{Q}})^{\mathrm{Br}\, X} \neq \varnothing.$$

In words, the local-global principle for integral points holds if and only if *there is no Brauer–Manin obstruction. If an integral solution does not exist, the Brauer–Manin obstruction is the unique reason for this failure.*

To prove this we need two deep ingredients: the strong approximation property for G, and a cohomological description of the Brauer–Manin pairing in terms of the finite abelian group $\mathrm{Pic}\, H$.

4.1 Ingredients

Theorem 4.2 (Strong approximation for spin groups, Kneser, Platonov–Rapinchuk)**.** *Let G be a semisimple simply connected algebraic group over $\mathbb{Q}$. Let S_0 be a finite set of places of $\mathbb{Q}$ such that for every almost simple factor G' of G, there exists a place $v \in S_0$ with $G'(\mathbb{Q}_v)$ non-compact. Then G satisfies* strong approximation *with respect to S_0: the diagonal image of $G(\mathbb{Q})$ is dense in the set of S_0-adeles of G. Concretely, for any finite set $S_2 \supset S_0$, any elements $\{g_v\} \in \prod_{v \in S_2 \setminus S_0} G(\mathbb{Q}_v)$ can be approximated arbitrarily closely (in the v-adic topology) by a global element $g \in G(\mathbb{Q})$ which is integral at all places outside S_2.*

In our situation, $G = \mathrm{Spin}(f)$ is semisimple simply connected. Since f is indefinite, there exists a real place ∞ (or at least one archimedean place) where f is isotropic; then $G(\mathbb{R})$ is non-compact. Hence we may take $S_0 = \{\infty\}$ and obtain strong approximation away from the archimedean place.

Theorem 4.3 (Tate–Nakayama for connected groups, Kottwitz; [2] Theorem 3.1)**.** *Let H be a connected linear algebraic group over $\mathbb{Q}$. For each place v of $\mathbb{Q}$, there is a natural pairing*

$$H^1(\mathbb{Q}_v, H) \times \mathrm{Pic}\, H_{\mathbb{Q}_v} \to \mathrm{Br}\, \mathbb{Q}_v \simeq \mathbb{Q}/\mathbb{Z}$$

induced by the cup-product and the isomorphism $\mathrm{Pic}\, H \simeq \mathrm{Ext}^c(H, \mathbb{G}_m)$. The induced maps fit into an exact sequence of pointed sets

$$H^1(\mathbb{Q}, H) \to \bigoplus_v H^1(\mathbb{Q}_v, H) \to \mathrm{Hom}(\mathrm{Pic}\, H, \mathbb{Q}/\mathbb{Z}),$$

where the Hom group is finite because $\mathrm{Pic}\, H$ is finite.

Theorem 4.4 (Exact sequence of points on homogeneous spaces)**.** *Let G be a connected linear algebraic group over $\mathbb{Q}$, $H \subset G$ a connected subgroup, and $X = G/H$. Choose a finite set S of places (large enough for (G, H)). Then there is a commutative diagram with exact rows and columns:*

$$\begin{array}{ccc}
 & & G(\mathbb{Q}) \longrightarrow G(_{\mathbb{Q}}) \\
 & & \downarrow \\
X(\mathbb{Q}) \longrightarrow X(_{\mathbb{Q}}) & \longrightarrow & \mathrm{Hom}(\mathrm{Br}\, X/\mathrm{Br}\, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^1(\mathbb{Q}, H) \longrightarrow \bigoplus_v H^1(\mathbb{Q}_v, H) & \longrightarrow & \mathrm{Hom}(\mathrm{Pic}\, H, \mathbb{Q}/\mathbb{Z}) \\
\downarrow & & \downarrow \\
H^1(\mathbb{Q}, G) \longrightarrow \bigoplus_v H^1(\mathbb{Q}_v, G). & &
\end{array}$$

Moreover, if G is semisimple simply connected, the right vertical map $\mathrm{Hom}(\mathrm{Br}\, X/\mathrm{Br}\, \mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(\mathrm{Pic}\, H, \mathbb{Q}/\mathbb{Z})$ *is an isomorphism, and the kernel of* $X(\mathbb{A}_{\mathbb{Q}}) \to \mathrm{Hom}(\mathrm{Br}\, X/\mathrm{Br}\, \mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ *coincides with the kernel of the composite map* $X(\mathbb{A}_{\mathbb{Q}}) \to \bigoplus_v H^1(\mathbb{Q}_v, H) \to \mathrm{Hom}(\mathrm{Pic}\, H, \mathbb{Q}/\mathbb{Z})$.

This theorem is a combination of Sansuc's exact sequence, the isomorphism (3) of Section 3.2, and the commutative diagram of Proposition 2.9 in [2]. The last statement follows because $\mathrm{Br}\, X/\mathrm{Br}\, \mathbb{Q} \cong \mathrm{Pic}\, H$ via the map induced by the H-torsor $G \to X$.

Theorem 4.5 (Integral version, [2] Theorem 3.7)**.** *Let G be semisimple simply connected, $H \subset G$ a connected subgroup, $X = G/H$. Let S_0 be as in Theorem 4.2. Let S be a finite set of places containing all archimedean places and large enough for (G, H) and a suitable integral model $\mathbf{X}$ of X (so that $\mathbf{X} \times_{\mathbb{Z}} \mathbb{Q} \cong X$). Then the conditions "$\{M_v\} \in \prod_{v\in S} X(\mathbb{Q}_v) \times \prod_{v\notin S} \mathbf{X}(\mathbb{Z}_v)$ is orthogonal to $\mathrm{Br}\, X$" and "its projection to S lies in the kernel of $\prod_{v\in S} X(\mathbb{Q}_v) \to \mathrm{Hom}(\mathrm{Pic}\, H, \mathbb{Q}/\mathbb{Z})$" are equivalent. Moreover, if such a point exists, then for any finite set $S_2 \supset S_0$ there exists $M \in \mathbf{X}(\mathbb{Z}_{S_0})$ (global S_0-integral point) arbitrarily close to the given local points outside S_0. In particular $\mathbf{X}(\mathbb{Z}_{S_0}) \neq \varnothing$.*

4.2 Sketch of proof of Theorem 4.1

We now assemble the ingredients.

Step 1: Reduction to the Brauer–Manin condition. The diagonal inclusion $\mathbf{X}(\mathbb{Z}) \hookrightarrow \mathbf{X}(\mathbb{A}_{\mathbb{Q}})$ combined with the global reciprocity law implies that any integral point belongs to the Brauer–Manin set. Hence $\mathbf{X}(\mathbb{Z}) \neq \varnothing \Rightarrow \mathbf{X}(\mathbb{A}_{\mathbb{Q}})^{\mathrm{Br}\, X} \neq \varnothing$ is automatic. The non-trivial direction is the converse.

Step 2: Cohomological interpretation of the Brauer–Manin condition. Assume there exists an adelic family $\{M_v\} \in \mathbf{X}(\mathbb{A}_{\mathbb{Q}})$ such that $\langle \{M_v\}, \alpha \rangle = 0$ for all $\alpha \in \mathrm{Br}\, X$. By the isomorphism $\mathrm{Br}\, X/\mathrm{Br}\, \mathbb{Q} \cong \mathrm{Pic}\, H$ of Section 3.2, this is equivalent to $\{M_v\}$ being in the kernel of the map

$$X(\mathbb{A}_{\mathbb{Q}}) \to \mathrm{Hom}(\mathrm{Pic}\, H, \mathbb{Q}/\mathbb{Z})$$

induced by the pairing of Theorem 4.3. Indeed, the exact sequence of Theorem 4.4 shows that for a semisimple simply connected G, the two kernels coincide.

Step 3: From adeles to a set of local points with finite constraints. Take a finite set S as in Theorem 4.5. Because the family $\{M_v\}$ lies in the kernel of the above map, its restriction to S lies in the kernel of the composite

$$\prod_{v\in S} X(\mathbb{Q}_v) \to \prod_{v\in S} H^1(\mathbb{Q}_v, H) \to \mathrm{Hom}(\mathrm{Pic}\, H, \mathbb{Q}/\mathbb{Z}).$$

(The contribution of places outside S vanishes automatically because for $v \notin S$, $\mathbf{X}(\mathbb{Z}_v)$ maps into the trivial subset of $H^1(\mathbb{Q}_v, H)$ by Lang's theorem and Hensel's lemma.)

Step 4: Existence of a global rational point and strong approximation. Using Theorem 4.4 again, the fact that the restriction $\{M_v\}_{v\in S}$ is orthogonal to $\mathrm{Pic}\, H$ implies that there exist a global rational point $N \in X(\mathbb{Q})$ and an adele $\{g_v\} \in G(\mathbb{A}_{\mathbb{Q}})$ such that $g_v \cdot N = M_v$ for every place v. This is exactly the conclusion of Theorem 3.3 in [2],

which follows from diagram chasing and the surjectivity of the map $\bigoplus_v H^1(\mathbb{Q}_v, H) \to \mathrm{Hom}(\mathrm{Pic}\, H, \mathbb{Q}/\mathbb{Z})$ provided by Theorem 4.3.

Now apply strong approximation (Theorem 4.2) to the group $G = \mathrm{Spin}(f)$ with $S_0 = \{\infty\}$ (recall that f is indefinite, so $G(\mathbb{R})$ is non-compact). We obtain a global element $g \in G(\mathbb{Q})$ that is arbitrarily close to g_v for all $v \in S_2 \setminus S_0$ (where S_2 contains S and any additional finite set) and integral at all other finite places.

Step 5: Construction of an everywhere integral point. Define $M = g \cdot N \in X(\mathbb{Q})$. For each finite place $v \notin S_2$, the element g is integral and N is rational, but a priori M need not be in $\mathbf{X}(\mathbb{Z}_v)$. However, because g is very close to g_v for $v \in S_2 \setminus S_0$, and g_v sends N to $M_v \in \mathbf{X}(\mathbb{Z}_v)$, an open neighbourhood argument (the set $\mathbf{X}(\mathbb{Z}_v)$ is open in $X(\mathbb{Q}_v)$) guarantees that M actually belongs to $\mathbf{X}(\mathbb{Z}_v)$ for all $v \notin S_0$. By construction M also belongs to $\mathbf{X}(\mathbb{Z}_v)$ for $v \notin S_2 \cup S_0$. Hence M is integral at all places except possibly $S_0 = \{\infty\}$; but the real place is archimedean, and by definition $\mathbf{X}(\mathbb{Z}_\infty) = X(\mathbb{R})$, so M is an *everywhere* integral point: $M \in \mathbf{X}(\mathbb{Z})$. This completes the proof of $\mathbf{X}(_{\mathbb{Q}})^{\mathrm{Br}\, X} \neq \varnothing \Rightarrow \mathbf{X}(\mathbb{Z}) \neq \varnothing$.

Remarks.

- In the case $m \geq 4$, we have $\mathrm{Pic}\, H = 0$, hence $\mathrm{Br}\, X/\mathrm{Br}\, \mathbb{Q} = 0$. The Brauer–Manin condition is vacuous, and the theorem reduces to the well-known fact that for indefinite quadratic forms of rank ≥ 4, the Hasse principle for integral representations holds (see Theorem 6.1 in [2]).
- When $m = 3$ and d is a square, again $\mathrm{Pic}\, H = 0$ and there is no obstruction. When d is not a square, the group $\mathrm{Pic}\, H \cong \mathbb{Z}/2\mathbb{Z}$ gives a single nontrivial Brauer class that must be evaluated. The examples of Section 5 illustrate exactly this mechanism.
- The indefiniteness is used solely to obtain a real place with non-compact $G(\mathbb{Q}_v)$; without it strong approximation would not be available and the theorem would fail, as shown in Section 6 for a positive definite form.

5 Example 1: an indefinite form with obstruction

We treat in detail the example $x^2 - 2y^2 + 64z^2 = m^2$, where m is a positive integer with $m \equiv \pm 3 \pmod 8$. This originates from Cassels [1] and is analyzed in [2] §8.1.1. The quadratic form $f(x, y, z) = x^2 - 2y^2 + 64z^2$ is indefinite (signature $(2, 1)$). We consider primitive solutions, so we remove the origin from the equation; this does not affect the Brauer group arguments.

5.1 Local solubility

It is easy to check that the equation has primitive solutions in every $\mathbb{Z}_p$ and in $\mathbb{R}$. For odd primes one uses Hensel's lemma, for $p = 2$ a direct check works, and the real locus is clearly nonempty. Hence

$$\prod_v \mathbf{X}(\mathbb{Z}_v) \neq \varnothing.$$

5.2 The Brauer group of X

Write the equation as

$$x^2 - 2y^2 = m^2 - 64z^2 = (m + 8z)(m - 8z).$$

A direct computation using the method of §5.8 of [2] shows that the quaternion algebra

$$\alpha = (m + 8z, 2) \in \operatorname{Br} \mathbb{Q}(X)$$

extends to an element of $\operatorname{Br} X$ and its class generates the quotient $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$, which is of order 2. Equivalently one may use $(m - 8z, 2)$ because on X

$$(m + 8z, 2) = (m + 8z, -2(m - 8z)) = (m + 8z, m - 8z) = (m - 8z, 2).$$

The construction starts from the rational point $M = (\frac{m}{2}, \frac{m}{2}, 0)$ (when m is odd). The tangent hyperplane at the corresponding projective point on $x^2 - 2y^2 + 64z^2 - m^2t^2 = 0$ is $2x - 2y - mt = 0$, which after dehomogenization yields $\rho = x - y - \frac{m}{2}$. The associated quaternion algebra is $(\rho, -m^2 \cdot \operatorname{disc}(f))$; a short computation identifies its class with $(m + 8z, 2)$.

5.3 Evaluation at local integral points

For an adelic point $(P_v) \in \prod \mathbf{X}(\mathbb{Z}_v)$ we compute the Brauer–Manin sum

$$\sum_v \operatorname{inv}_v \alpha(P_v) \in \mathbb{Q}/\mathbb{Z}.$$

Real place $v = \infty$. Over $\mathbb{R}$ the Hilbert symbol $(a, 2)_\infty$ is trivial for every real number a. Hence $\operatorname{inv}_\infty \alpha(P_\infty) = 0$.

Odd primes $p \neq 2$. Let $P = (x, y, z) \in \mathbf{X}(\mathbb{Z}_p)$. Because the solution is primitive, one shows that the p-adic valuation of $m + 8z$ is even, or that $m + 8z$ is a norm from the unramified extension $\mathbb{Q}_p(\sqrt{2})/\mathbb{Q}_p$ when 2 is not a square modulo p. In all cases $(m + 8z, 2)_p = 0$. Consequently

$$\operatorname{inv}_p \alpha(P) = 0 \qquad (p \neq 2).$$

Dyadic place $p = 2$. Now let $P = (x, y, z) \in \mathbf{X}(\mathbb{Z}_2)$. Since $m \equiv \pm 3 \pmod 8$, we have $m + 8z \equiv m \pmod 8$, so $m + 8z \equiv 3$ or $5 \pmod 8$. Both are non-square units in $\mathbb{Q}_2$ whose Hilbert symbol with 2 is $\frac{1}{2}$; indeed in $\mathbb{Q}_2$,

$$(3, 2)_2 = \frac{1}{2}, \qquad (5, 2)_2 = \frac{1}{2}.$$

Thus for every primitive 2-adic solution,

$$\operatorname{inv}_2 \alpha(P) = \frac{1}{2} \neq 0.$$

5.4 The Brauer–Manin obstruction

Summing the local invariants over all places we obtain

$$\sum_v \operatorname{inv}_v \alpha(P_v) = 0(\infty) + \sum_{p\neq 2} 0 + \frac{1}{2}\,(p=2) = \frac{1}{2} \in \mathbb{Q}/\mathbb{Z}.$$

Hence the Brauer–Manin set

$$\Big(\prod_v \mathbf{X}(\mathbb{Z}_v)\Big)^{\operatorname{Br} X} = \varnothing,$$

and by Theorem 4.1 there is no primitive integer solution. The obstruction is completely captured by the single algebra α.

6 Example 2: a positive definite form and the necessity of indefiniteness

We now examine the equation

$$x^2 + y^2 + 10z^2 = 3.$$

The quadratic form $x^2 + y^2 + 10z^2$ is positive definite (signature $(3,0)$). Therefore the general theory of Section 4 does *not* apply; we will see that there is no Brauer–Manin obstruction, yet the equation has no integer solutions.

6.1 Local solubility

One checks easily that the equation has solutions in every $\mathbb{Z}_p$ and in $\mathbb{R}$:

- $\mathbb{R}$: obviously nonempty.
- Odd primes: Hensel's lemma applies to the smooth $\mathbb{F}_p$-variety.
- $p = 2$: For example, $(x, y, z) = (1, 0, 1)$ gives $1 + 0 + 10 = 11 \equiv 3 \pmod 8$ but one can find actual $\mathbb{Z}_2$-points (see the draft for details).

Thus $\mathbf{X}(_{\mathbb{Q}}) \neq \varnothing$.

6.2 An Azumaya algebra on X

Rewrite the equation as

$$x^2 + y^2 = 3 - 10z^2.$$

The left-hand side is a norm from the quadratic extension $\mathbb{Q}(i)/\mathbb{Q}$: $x^2+y^2 = N_{\mathbb{Q}(i)/\mathbb{Q}}(x+iy)$. This suggests the quaternion algebra

$$B = (-1, x^2 + y^2) = (-1, 3 - 10z^2) \in \operatorname{Br}\mathbb{Q}(X). \tag{1}$$

Proposition 6.1. *The algebra B extends to an Azumaya algebra on the whole affine surface X.*

Proof. The only possible ramification divisor is the locus where the second entry vanishes, $D : 3 - 10z^2 = 0$. Let η be the generic point of an irreducible component of D. On the local ring $\mathcal{O}_{X,\eta}$, $3 - 10z^2$ is a uniformiser. The residue of B at D is the class of the first entry -1 modulo squares in the residue field $\kappa(D)$:

$$\partial_D(B) = -1 \pmod{\kappa(D)^{\times 2}}.$$

On D the defining equation forces $x^2 + y^2 = 3 - 10z^2 = 0$. Hence in the function field $\kappa(D)$ we have $(x/y)^2 = -1$ (if $y \neq 0$ on this component). Therefore $\sqrt{-1} \in \kappa(D)$, which means that -1 is a square in $\kappa(D)$. Consequently $\partial_D(B) = 1$, i.e. the residue is trivial. Thus B is unramified everywhere and defines an element of $\operatorname{Br} X$. □

6.3 Evaluation of local invariants

Take an arbitrary adelic point $(P_v) \in \mathbf{X}(_{\mathbb{Q}})$. For a place v we compute

$$\operatorname{inv}_v(B(P_v)) = (-1, 3 - 10z^2)_v \in \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}.$$

Real place $v = \infty$. From the defining equation, $3 - 10z^2 = x^2 + y^2 \geq 0$ (and > 0 on a dense open set). For any positive real number $t > 0$, the Hilbert symbol $(-1, t)_\infty = 1$. Thus for every real point P_∞,

$$\operatorname{inv}_\infty(B(P_\infty)) = 0.$$

$p = 2$. Work modulo 8. Since $10 \equiv 2 \pmod 8$, the equation reduces to

$$x^2 + y^2 + 2z^2 \equiv 3 \pmod 8.$$

If z is even, then $2z^2 \equiv 0 \pmod 8$, which would require $x^2 + y^2 \equiv 3 \pmod 8$. But a sum of two squares mod 8 can only be $0, 1, 2, 4, 5$. Hence z must be odd. With z odd, $z^2 \equiv 1 \pmod 8$, so

$$3 - 10z^2 \equiv 3 - 10 \equiv -7 \equiv 1 \pmod 8.$$

Any element of $\mathbb{Z}_2^\times$ congruent to 1 mod 8 is a square (Hensel's lemma). Therefore $3 - 10z^2$ is a square in $\mathbb{Z}_2$, and the Hilbert symbol $(-1, \text{square})_2 = 1$. Hence

$$\operatorname{inv}_2(B(P_2)) = 0.$$

Odd primes $p > 2$. We distinguish two cases.

Case 1: $p \nmid (3 - 10z^2)$. Then both -1 and $3 - 10z^2$ are units in $\mathbb{Z}_p$. For an odd prime, the Hilbert symbol of two units is always 1. Thus $\operatorname{inv}_p = 0$.

Case 2: $p \mid (3 - 10z^2)$. From the equation we obtain $x^2 + y^2 \equiv 0 \pmod p$.

- If $p \equiv 1 \pmod 4$, then -1 is already a square in $\mathbb{Z}_p$, so $(-1, \cdot)_p = 1$.
- If $p \equiv 3 \pmod 4$, then -1 is not a square mod p. The congruence $x^2 + y^2 \equiv 0 \pmod p$ forces $x \equiv y \equiv 0 \pmod p$. Hence the p-adic valuation $v_p(x^2 + y^2)$ is at least 2, and in particular it is even. Since $3 - 10z^2 = x^2 + y^2$, its p-adic valuation is even. The Hilbert symbol formula gives

$$(-1, 3 - 10z^2)_p = \left(\frac{-1}{p}\right)^{v_p(3-10z^2)} = 1.$$

In all subcases, $\operatorname{inv}_p = 0$.

Thus for every odd prime p, $\operatorname{inv}_p(B(P_p)) = 0$.

6.4 Conclusion: no Brauer–Manin obstruction, yet no integer solution

Putting all local invariants together,

$$\sum_{v} \mathrm{inv}_v(B(P_v)) = 0 + 0 + 0 = 0.$$

The sum is identically zero for any family of local integral points. Consequently the Brauer–Manin set is the whole adelic space:

$$\mathbf{X}(\mathbb{Q})^{\mathrm{Br}\, X} = \mathbf{X}(\mathbb{Q}) \neq \varnothing.$$

There is no integral Brauer–Manin obstruction arising from B. However, a simple check (e.g. modulo 4 or modulo 8) shows that the equation $x^2 + y^2 + 10z^2 = 3$ has absolutely no integer solutions. Indeed, working modulo 4, $x^2 + y^2 + 2z^2 \equiv 3 \pmod 4$ is impossible because the left-hand side can only be $0, 1, 2$. This demonstrates that for positive definite forms, the Brauer–Manin obstruction may vanish while other arithmetic obstructions (coming from the class group or the genus) prevent the existence of integral points. The indefiniteness assumption in Theorem 4.1 is therefore essential.

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