Final report: classification of simple Lie algebra

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Chapter 1

Root systems

1.1 Abstract root system

Definition 1.1.1. *A **root system** (E, R) is a finite-dimensional real vector space E with an inner product $\langle \cdot, \cdot \rangle$, together with a finite collection R of nonzero vectors in E satisfying the following properties:*

1. *The vectors in R span E.*
2. *If $\alpha \in R$, then $c\alpha \in R$ if and only if $c = 1$ or $c = -1$.*
3. *If $\alpha, \beta \in R$, then so is $S_\alpha \cdot \beta$, where S_α is the linear transformation of E defined by*

$$S_\alpha \cdot \beta = \beta - 2 \cdot \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad \beta \in E.$$

4. *For $\alpha, \beta \in R$, the quantity $2 \cdot \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ is an integer.*

*The dimension of E is called **the rank of the root system**, and the elements of R are called **roots**.*

Definition 1.1.2. *If (E, R) is a root system, the **Weyl group** W of R is the subgroup of the orthogonal group of E generated by the reflections $S_\alpha, \alpha \in R$.*

Since for $\forall \beta \in R$, $\beta = S_\alpha \cdot (S_\alpha \cdot \beta)$, each S_α maps R onto itself. It follows that every element of W maps R onto itself. Thus, the Weyl group is a finite subgroup of $O(E)$ and may be regarded as a subgroup of the permutation group on R.

We denote the action of $w \in W$ on $H \in E$ by $w \cdot H$.

Proposition 1.1.3. *Suppose (E, R) and (F, S) are root systems. Consider the vector space $E \oplus F$, with the natural inner product determined by the inner products on E and F. Then $R \cup S$ is a root system in $E \oplus F$, called the direct sum of R and S.*

Proof. Check it by def. □

Definition 1.1.4. *A root system (E, R) is called* ***reducible*** *if there exists an orthogonal decomposition $E = E_1 \oplus E_2$ with $\dim E_1 > 0$ and $\dim E_2 > 0$ such that every element of R is either in E_1 or E_2.*

If no such decomposition exists, (E, R) is called ***irreducible****.*

Definition 1.1.5. *Two root systems (E, R) and (F, S) are said to be* ***isomorphic*** *if there exists an invertible linear transformation $A : E \to F$ such that A maps R onto S and such that for $\forall \alpha \in R, H \in E$, we have*

$$A(S_\alpha \cdot H) = S_{A(\alpha)} \cdot (A(H)).$$

A map A with this property is called an ***isomorphism****.*

Note that the linear map A is not required to preserve the inner product, but only to preserve the reflection about the roots.

We now establish a basic result limiting the possible angles and length ratios occurring in a root system.

Proposition 1.1.6. *Suppose α and β are roots, α is not a multiple of β, and $\langle \alpha, \alpha \rangle \geq \langle \beta, \beta \rangle$. Then one of the following holds:*

1. $\langle \alpha, \beta \rangle = 0$
2. $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$ *and the angle between α and β is $\frac{\pi}{3}$ or $\frac{2\pi}{3}$*
3. $\langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$ *and the angle between α and β is $\frac{\pi}{4}$ or $\frac{3\pi}{4}$*
4. $\langle \alpha, \alpha \rangle = 3\langle \beta, \beta \rangle$ *and the angle between α and β is $\frac{\pi}{6}$ or $\frac{5\pi}{6}$.*

Proof. If $\alpha, \beta \in R$ and $m_1 = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$, $m_2 = \frac{2\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle}$, then m_1 and m_2 are integers.

Assume $\langle \alpha, \alpha \rangle \geq 7\langle \beta, \beta \rangle$ and note that

$$m_1 m_2 = 4 \frac{\langle \alpha, \beta \rangle^2}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} = 4 \cos^2 \theta,$$

where θ is the angle between α and β, and that

$$\frac{m_2}{m_1} = \frac{\langle \alpha, \alpha \rangle}{\langle \beta, \beta \rangle} \geq 1$$

whenever $\langle \alpha, \beta \rangle \neq 0$.

Since $m_1, m_2 \in \mathbb{Z}$, $m_1 m_2 \geq 0$, we have $0 \leq m_1 m_2 \leq 4$.

1. $m_1 m_2 = 0 \implies \langle \alpha, \beta \rangle = 0 \implies \alpha$ and β are orthogonal.
2. $m_1 m_2 = 4 \implies \cos^2 \theta = 1 \implies \alpha$ and β are multiples.
3. $m_1 m_2 = 1 \implies \cos^2 \theta = \frac{1}{4} \implies \cos \theta = \pm \frac{1}{2} \implies \theta = \frac{\pi}{3}, \frac{2\pi}{3}$ and $m_1 = m_2$. Thus, $\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$.
4. $m_1 m_2 = 2 \implies \cos^2 \theta = \frac{1}{2} \implies \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ and $m_2 = 2m_1 \implies \langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$.

5. $m_1 m_2 = 3 \implies \cos^2\theta = \frac{3}{4} \implies \theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ and $m_2 = 3m_1 \implies \langle\alpha,\alpha\rangle = 3\langle\beta,\beta\rangle$.
   Then, we proved this proposition.

□

By this proposition, we can make some inferences about the relationship between the roots :

Corollary 1.1.7. *Suppose α and β are roots. If the angle between α and β is strictly obtuse, then $\alpha+\beta$ is a root. If the angle between α and β is strictly acute, then $\alpha-\beta$ and $\beta-\alpha$ are also roots.*

Proof. Without loss of generality: $\langle\alpha,\alpha\rangle \geq 7\langle\beta,\beta\rangle$.

1. If the angle θ is obtuse, then $S_\alpha\cdot\beta = \beta - 2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha = \beta - m_1\alpha$ with $m_1 < 0$. Thus, $S_\alpha\cdot\beta = \beta+\alpha$ is a root (since $m_1 = -1$ when θ is obtuse).
2. If the angle θ is acute, then $S_\alpha\cdot\beta = \beta - m_1\alpha = \beta-\alpha$ ($m_1 = 1$) is a root. Thus, $(-1)\cdot(\beta-\alpha) = \alpha-\beta$ is a root.

□

1.2 Examples in Rank 2

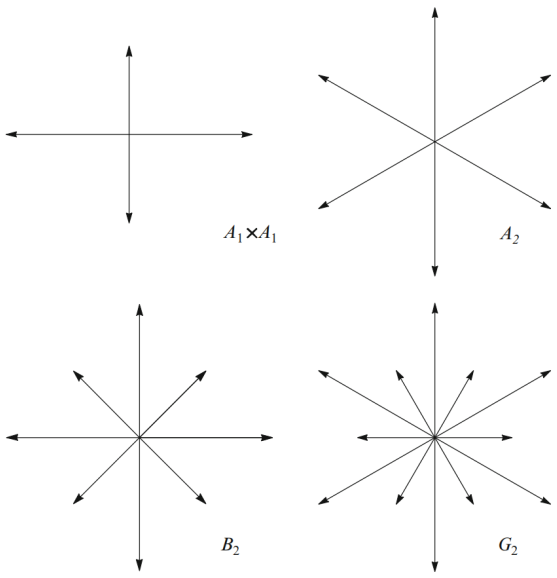


Figure 1.1: The rank-two root systems

Proposition 1.2.1. *Every rank-two root system is isomorphic to one of the systems above.*

Proof. Without loss of generality, assume $E = \mathbb{R}^2$ and $R \subset \mathbb{R}^2$ is a root system. Let θ be the smallest angle occurring between any two vectors in R. Since R spans $\mathbb{R}^2$, we can find two linearly independent $\alpha, \beta \in R$. If the angle between α and β is $> \frac{\pi}{2}$, then the angle between α and $-\beta$ is $< \frac{\pi}{2}$, implying $0 < \theta < \frac{\pi}{2}$. By Proposition 8.6, θ can only be $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, or $\frac{\pi}{6}$.

Let $\alpha, \beta \in R$ such that the angle between them is θ. Then:

$$-S_\beta \cdot \alpha = -\left(\alpha - 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}\beta\right) = -\alpha + 2\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle}\beta$$

This implies $\alpha + (-S_\beta \cdot \alpha)$ is a multiple of β. Hence, $-S_\beta \cdot \alpha$ is a vector at angle θ to β but on the opposite side of β from α. The angle between α and $-S_\beta \cdot \alpha$ is 2θ. Similarly, $-S_\beta \cdot (-S_\beta \cdot \alpha)$ is at angle 3θ to α. Repeating this process generates vectors at angles $n\theta$ to α for all $n \in \mathbb{N}$. Since θ divides 2π for each allowed value of θ, the roots must form a finite system. Additional roots would contradict θ being the smallest angle.

- $\theta = \frac{\pi}{2} \implies A_1 \times A_1$
- $\theta = \frac{\pi}{3} \implies A_2$ ($\langle \alpha, \alpha \rangle = \langle \beta, \beta \rangle$)
- $\theta = \frac{\pi}{4} \implies B_2$ ($\langle \alpha, \alpha \rangle = 2\langle \beta, \beta \rangle$)
- $\theta = \frac{\pi}{6} \implies G_2$ ($\langle \alpha, \alpha \rangle = 3\langle \beta, \beta \rangle$)

□

Proposition 1.2.2. *If R is a rank-two root system with minimum angle $\theta = \dfrac{2\pi}{n}$, $n = 4, 6, 8, 12$, then the Weyl group of R is the symmetry group of a regular $\dfrac{n}{2}$-gon ($D_{\frac{n}{2}}$).*

1.3 Duality

Definition 1.3.1. *If (E, R) is a root system, then for each root $\alpha \in R$, the **coroot** H_α is the vector given by*

$$H_\alpha = 2 \cdot \frac{\alpha}{\langle \alpha, \alpha \rangle}.$$

*The set of all coroots is denoted $R^\vee$ and is called the **dual root system** to R.*

Proposition 1.3.2. *If R is a root system, then $R^\vee$ is also a root system and the Weyl group for $R^\vee$ is the same as the Weyl group for R. Furthermore, $(R^\vee)^\vee = R$.*

Proof. We have

$$\langle H_\alpha, H_\alpha \rangle = 4 \cdot \frac{4}{\langle \alpha, \alpha \rangle} \implies H_{H_\alpha} = \frac{2H_\alpha}{\langle H_\alpha, H_\alpha \rangle} = \alpha \implies (R^\vee)^\vee = R.$$

Also, for $\forall \alpha, \beta \in R$,

$$\frac{2}{\langle H_\alpha, H_\alpha \rangle} = \frac{2}{\langle \alpha, \alpha \rangle} \cdot \frac{4}{\langle \alpha, \alpha \rangle \langle \beta, \beta \rangle} \cdot \langle \alpha, \beta \rangle = 2 \cdot \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle},$$

which means

$$2 \cdot \frac{\langle H_\alpha, H_\beta \rangle}{\langle H_\beta, H_\beta \rangle} \text{ is an integer.}$$

Furthermore,

$$S_{H_\alpha}(H) = H - 2 \cdot \frac{\langle \alpha, H \rangle}{\langle \alpha, \alpha \rangle} H_\alpha = S_\alpha(H).$$

Since S_α is an orthogonal transformation, we have

$$S_\alpha(H_\beta) = 2 \cdot \frac{S_\alpha(\beta)}{\langle \beta, \beta \rangle} = 2 \cdot \frac{S_\alpha(\beta)}{\langle S_\alpha(\beta), S_\alpha(\beta) \rangle} = H_{S_\alpha(\beta)},$$

which means $R^\vee$ is invariant under each reflection S_{H_α} (i.e., S_α). Thus, $R^\vee$ is a root system with the same Weyl group as R.

□

Note that

$$\frac{\langle H_\alpha, H_\beta \rangle}{\langle H_\beta, H_\beta \rangle} = \frac{\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle} \quad (\forall \alpha, \beta \in R) \implies \frac{\langle H_\alpha, H_\beta \rangle}{\langle H_\alpha, H_\alpha \rangle} = \frac{\langle \beta, \beta \rangle}{\langle \alpha, \alpha \rangle}.$$

If all roots in R have the same length, then $R^\vee$ is isomorphic to R. If not all roots in R have the same length, $R^\vee$ may or may not be isomorphic to R.

1.4 Bases and Weyl Chambers

Definition 1.4.1. *If (E, R) is a root system, a subset Δ of R is called a **base** if the following conditions hold:*

1. *Δ is a basis for E as a vector space.*
2. *Each root $\alpha \in R$ can be expressed as a linear combination of elements of Δ with integer coefficients, and in such a way that the coefficients are either all non-negative or all non-positive.*

The roots for which the coefficients are non-negative are called **positive roots** and the others are called **negative roots** (relative to the base Δ). The set of positive roots relative to a fixed base Δ is denoted R^+ and the set of negative roots is denoted R^-. The elements of Δ are called the **positive simple roots**.

Definition 1.4.2. *The set of positive roots relative to a fixed base Δ is denoted R^+ and the set of negative roots is denoted R^-. The elements of Δ are called the **positive simple roots**.*

Proposition 1.4.3. *If α and β are distinct elements of a base Δ for R, then $\langle \alpha, \beta \rangle \leq 0$.*

Proof. We have $\alpha \neq \beta$. If $\langle \alpha, \beta \rangle > 0$, then the angle between α, β is acute, implying $\alpha - \beta \in R$. By the uniqueness of the base of the vector space E, this leads to a contradiction on condition 2. □

Proposition 1.4.4. *Let V be a hyperplane through the origin in E that does not contain any element of R.*

Proof. We have that for any finite number of proper subspace of a vector space E, their union cannot be E. Then, for a root system (E, R), let V_α denote the hyperplane

$$V_\alpha = \{H \in E \mid \langle \alpha, H \rangle = 0\}.$$

Then, there exists $H \in E$ such that $H \notin V_\alpha$ for any $\alpha \in R$. Let V be the hyperplane through the origin orthogonal to H. Since H is not in V_α, it follows that H is not orthogonal to any $\alpha \in R$, meaning no root α lies in V. □

Definition 1.4.5. *Let (E, R) be a root system. Let V be a hyperplane through the origin in E such that V does not contain any root. Choose one "side" of V and let R^* denote the set of roots on this side of V. An element $\alpha \in R^*$ is called* ***decomposable*** *if there exist $\beta, \gamma \in R^*$ such that*

$$\alpha = \beta + \gamma.$$

If no such decomposition exists, α is called ***indecomposable****.*

The "sides" of V are the connected components of $E \setminus V$. Equivalently, if H is a nonzero vector orthogonal to V, then $V = \{u \in E \mid \langle u, H \rangle = 0\}$. The two sides of V are the sets:

$$\{u \in E \mid \langle u, H \rangle > 0\} \quad \text{and} \quad \{u \in E \mid \langle u, H \rangle < 0\}.$$

Theorem 1.4.6. *Let (E, R) be a root system. Let V be a hyperplane through the origin in E such that V does not contain any root, and R^+ is the set of roots lying on a fixed side of V. Then, the set of indecomposable elements of R^+ is a base for R.*

Proof. Let Δ denote the set of all indecomposable elements in R^+. Choose a nonzero vector H orthogonal to V; the chosen side of V is $\{\mu \in E \mid \langle H, \mu \rangle > 0\}$.

Step I. WTS: Every $d \in R^+$ can be expressed as a linear combination of elements of Δ with non-negative integer coefficients.

If not, then among all elements of R^+ that cannot be expressed in this way, choose d such that $\langle H, d \rangle$ is minimal. Clearly, $d \notin \Delta$, so there exist $\beta_1, \beta_2 \in R^+$ such that $d = \beta_1 + \beta_2$, and neither β_1 nor β_2 can be expressed as such a combination. However, $\langle H, d \rangle = \langle H, \beta_1 \rangle + \langle H, \beta_2 \rangle > 0$, contradicting minimality.

Step II. WTS: If $d \neq \beta \in \Delta$, then $\langle d, \beta \rangle \leq 0$.

If $\langle d, \beta \rangle > 0$, then $d - \beta \in R$ or $\beta - d \in R$. Without loss of generality, assume $d - \beta \in R^+$. Then $d = (d - \beta) + \beta$, implying $\langle d, \beta \rangle \leq 0$.

Step III. WTS: The elements of Δ are linearly independent.

Suppose $\sum_{d \in \Delta} C_d d = 0$ for some coefficients C_d. Separate positive and negative coefficients:

$$\sum_{C_d > 0} C_d d = \sum_{C_\beta < 0} (-C_\beta)\beta.$$

Let $u = \sum_{C_d > 0} C_d d$. Then,

$$\langle u, u \rangle = \sum_{C_d > 0} \sum_{C_\beta < 0} C_d(-C_\beta)\langle d, \beta \rangle \leq 0,$$

by Step II. Hence $u = 0$. Since $\langle H, u \rangle = \sum C_d \langle H, d \rangle = 0$ and $\langle H, d \rangle > 0$, all $C_d = 0$.

Step IV. WTS: Δ is a base of (E, R).

It remains to show Δ spans E. Let $E' = \text{Span}(\Delta)$. Since $R = R^+ \cup R^-$ and $R^+ \subseteq E'$, the roots span E. Thus $E = E'$.

□

Theorem 1.4.7. *For any base Δ for R, there exists a hyperplane V and a side of U such that Δ arises as in Theorem 1.4.6.*

Proof. Let $\Delta = \{d_1, \ldots, d_r\}$ be a base for R. Since Δ is a base for E as a vector space, for any $c_1, \ldots, c_r \in \mathbb{R}$, there exists $T \in E$ such that $\langle T, d_i \rangle = c_i$. Choose T such that $\langle T, d_i \rangle > 0$ for all $d_i \in \Delta$.

Define the hyperplane:

$$U = \{H \in E \mid \langle T, H \rangle = 0\}.$$

For any root $d \in R^+$, we have $\langle T, d \rangle > 0$, and for any root $\beta \in R^-$, $\langle T, \beta \rangle < 0$. Thus, $\Delta \subseteq R^+$.

Suppose some $d \in \Delta$ is decomposable. Then there exist $\beta, \gamma \in R^+$ with $\beta \neq d$ such that $d = \beta + \gamma$. However, since β and γ can be expressed as linear combinations of $\Delta \setminus \{d\}$ (contradicting the indecomposability of d), all elements of Δ must be indecomposable. Hence, Δ is contained in the base relative to U.

Finally, since $|\Delta| = \dim E$ and the base relative to $U^\top$ also has cardinality $\dim E$, it follows that Δ is precisely the base relative to $U^\top$. □

Proposition 1.4.8. *If Δ is a base for R, then the set of ll coroots $H_\alpha, \alpha \in \Delta$ is a base for $R^\vee$*

Definition 1.4.9 (Open Weyl Chambers)**.** *The open Weyl Chambers for a root system (E, R) are the connected components of the complement of all root hyperplanes:*

$$E \setminus \bigcup_{\alpha \in R} V_\alpha,$$

where

$$V_\alpha = \{H \in E \mid \langle H, \alpha \rangle = 0\}$$

denotes the hyperplane through the origin orthogonal to α.

Definition 1.4.10 (Fundamental Weyl Chamber)**.** *Let $\Delta = \{d_1, \ldots, d_r\}$ be a base for R. The open fundamental Weyl chamber in E relative to Δ is the set of all $H \in E$ such that*

$$\langle d_j, H \rangle > 0 \quad \textit{for all } j = 1, \ldots, r.$$

Note that

1. The open fundamental Weyl chamber is convex (hence connected) and nonempty.
2. Since the only way one can exit afundamental Weyl chamber is passing through a point H where $\langle \alpha_j, H \rangle = 0$, then the open fundamental Weyl chamber is, indeed, an open Weyl chamber.

3. For any root α, the reflection S_α acts orthogonally on the hyperplane $V_\beta = \{H \in E \mid \langle H, \beta\rangle = 0\}$. Specifically, for all $H \in V_\beta$ and $a \in R$,

$$\langle H, \beta\rangle = \langle S_\alpha(H), S_\alpha(\beta)\rangle.$$

Consequently, for any $w \in W$ and any open Weyl chamber C, the image $w(C)$ is another open Weyl chamber

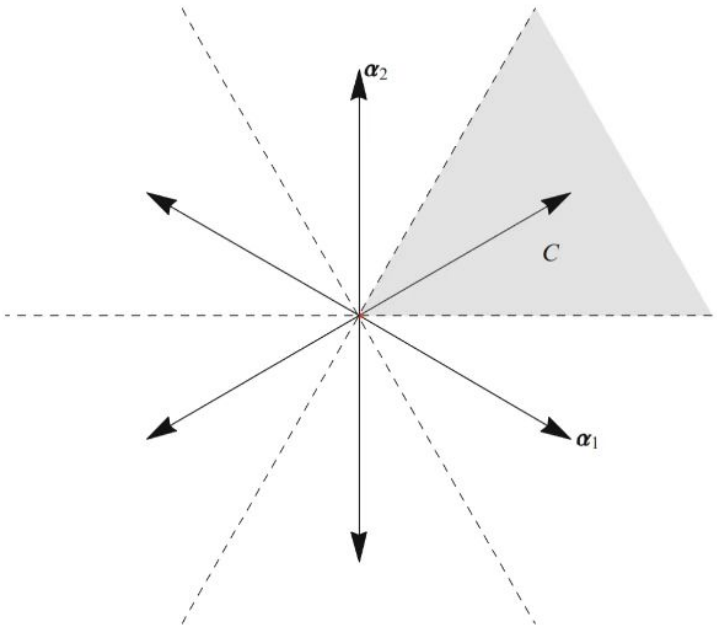


Figure 1.2: The shaded region C is the open fundamental Weyl chamber associated to the base α_1, α_2 for A_2

Proposition 1.4.11 (8.21)**.** *For each open Weyl chamber C, there exists a unique base Δ_C for R such that C is the open fundamental Weyl chamber associated to Δ_C. Moreover, the positive roots with respect to Δ_C are precisely those $\alpha \in R$ for which $\langle \alpha, H\rangle > 0$ for all $H \in C$.*

Proof. Let $H \in C$ and define the hyperplane $V = \{u \in E \mid \langle u, H\rangle = 0\}$. Since C is an open Weyl chamber, no root lies in V, i.e., $\langle \alpha, H\rangle \neq 0$ for all $\alpha \in R$. By Theorem 8.16, there exists a base $\Delta = \{d_1, \ldots, d_r\}$ such that all d_i lie on the same side of V as H, i.e.,

$$\langle d_i, H\rangle > 0 \quad \text{for } i = 1, \ldots, r.$$

For any positive root $\alpha \in R^+$ associated to Δ, we have $\langle \alpha, H\rangle > 0$, and for any negative root $\beta \in R^-$, $\langle \beta, H\rangle < 0$. Thus, C coincides with the open fundamental Weyl chamber defined by Δ.

To show uniqueness, suppose Δ' is another base whose associated fundamental Weyl chamber is C. Since Δ' must also lie on the same side of V as H, the positive systems R^+ of Δ and Δ' must coincide. Therefore, $\Delta' = \Delta$. □

Proposition 1.4.12. *Every root is an element of some base.*

1.5 Weyl Chamber and Weyl Group

Proposition 1.5.1. *The Weyl group acts transitively on the set of open Weyl chambers.*

Proof. Fix a base Δ, let C be the fundamental chamber associated to Δ, and let W' be the subgroup of W generated by $\{s_\alpha \mid \alpha \in \Delta\}$. Let D be any other chamber with $H \in C$ and $H' \in D$. Choose $w \in W'$ minimizing $|w \cdot H' - H|$. If $w \cdot H' \notin C$, there exists $\alpha \in \Delta$ such that $\langle \alpha, w \cdot H' \rangle < 0$. Then:

$$|w \cdot H' - H|^2 - |s_\alpha \cdot w \cdot H' - H|^2 = -4\langle \alpha, w \cdot H' \rangle \langle \alpha, H \rangle > 0,$$

implying $s_\alpha \cdot w \cdot H'$ is closer to H, contradicting minimality. Hence $w \cdot H' \in C$, proving transitivity. □

Proposition 1.5.2. *If Δ is a base, then W is generated by the reflections s_α with $\alpha \in \Delta$.*

Proof. We show $W' = W$. Let s_α be a reflection for any root α. By Proposition 1.4.12, $\alpha \in \Delta_D$ for some chamber D. Choose $w \in W'$ with $w \cdot D = C$, so $w \cdot \alpha \in \Delta$. Since:

$$s_{w \cdot \alpha} = w s_\alpha w^{-1},$$

we have $s_\alpha = w^{-1} s_{w \cdot \alpha} w$. As $w \cdot \alpha \in \Delta$, both w and $s_{w \cdot \alpha}$ lie in W'. Thus $s_\alpha \in W'$, implying $W = W'$. □

Lemma 1.5.3. *Let Δ be a base for R, C the associated fundamental Weyl chamber, and $w \in W$ with $w \neq I$. Suppose $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$ ($\alpha_j \in \Delta$) is a minimal expression for w. Then C and $w \cdot C$ lie on opposite sides of the hyperplane V_{α_1}.*

Proof. For $k = 1$, $w = s_{\alpha_1}$ flips C across V_{α_1}. Assume inductively the result holds for elements expressible with $k-1$ reflections. Let $w = s_{\alpha_1} \cdots s_{\alpha_k}$ and $u = s_{\alpha_1} \cdots s_{\alpha_{k-1}}$. By induction, $u \cdot C$ lies opposite C across V_{α_1}. If $w \cdot C$ were on the same side as C, then $u \cdot C$ and $w \cdot C = u s_{\alpha_k} \cdot C$ would lie on opposite sides of V_{α_1}, implying C and $s_{\alpha_k} \cdot C$ lie across $V_{u^{-1}\alpha_1}$.

Choosing $H \in \partial C \cap V_{\alpha_k}$, the transition $C \to s_{\alpha_k} \cdot C$ occurs only across V_{α_k}. Hence $V_{u^{-1}\alpha_1} = V_{\alpha_k}$, leading to:

$$s_{\alpha_k} = s_{u^{-1}\alpha_1} = u^{-1} s_{\alpha_1} u.$$

Substituting into $w = u s_{\alpha_k}$:

$$w = s_{\alpha_1} u = s_{\alpha_1}^2 s_{\alpha_2} \cdots s_{\alpha_{k-1}} = s_{\alpha_2} \cdots s_{\alpha_{k-1}},$$

contradicting the minimality of k. □

Proposition 1.5.4. *Let C be a Weyl chamber and let $H, H' \in \overline{C}$. If $w \cdot H = H'$ for some $w \in W$, then $H = H'$. In other words, two distinct elements of $\overline{C}$ cannot be in the same W-orbit.*

Proof. By induction on the minimal number k of reflections needed to express w. If $k = 0$, $w = I$ trivially satisfies $H' = H$. For $k > 0$, let $w = s_{\alpha_1} \cdots s_{\alpha_k}$. By Lemma 1.5.3, C and $w \cdot C$ lie on opposite sides of V_{α_1}, so:

$$(w \cdot \overline{C}) \cap \overline{C} \subset V_{\alpha_1}.$$

Thus $H' \in V_{\alpha_1}$. Applying s_{α_1}:

$$s_{\alpha_1} \cdot H' = H' \implies w' := s_{\alpha_1} w = s_{\alpha_2} \cdots s_{\alpha_k}.$$

Since w' has $k-1$ reflections, by induction $H' = H$. □

By the propositions above, we can easily get the following 3 propositions.

Proposition 1.5.5. *The Weyl group acts freely on the set of open Weyl chambers. If $H \in C$ (an open chamber) and $w \cdot H = H$ for some $w \in W$, then $w = I$.*

Proposition 1.5.6. *For any two bases Δ_1 and Δ_2 for R, there exists a unique $w \in W$ such that $w \cdot \Delta_1 = \Delta_2$.*

Proposition 1.5.7. *Let C be a Weyl chamber and $H \in E$. There exists exactly one point $H' \in \overline{C}$ such that $H' = w \cdot H$ for some $w \in W$.*

Proposition 1.5.8. *Let Δ be a base for R, R^+ the associated positive roots, and $\alpha \in \Delta$. If $\beta \in R^+$ and $\beta \neq \alpha$, then $s_\alpha \cdot \beta \in R^+$.*

Proof. Write $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma$ with $c_\gamma \geq 0$. Since $\beta \neq \alpha$, there exists $\gamma \neq \alpha$ with $c_\gamma > 0$. The reflection acts as:

$$s_\alpha \cdot \beta = \beta - n(\beta, \alpha)\alpha,$$

where $n(\beta, \alpha) \in \mathbb{Z}$. In the expansion of $s_\alpha \cdot \beta$, the coefficient of $\gamma \neq \alpha$ remains $c_\gamma > 0$. Since all coefficients in the expansion must be non-negative, $s_\alpha \cdot \beta$ is a positive root. □

1.6 Dykin Diagrams

Definition 1.6.1. *Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be a base for a root system R. The **Dynkin diagram** of R is a graph with:*

- *Vertices $v_1, \ldots, v_r$ corresponding to $\alpha_1, \ldots, \alpha_r$;*
- *Edges between distinct v_j and v_k determined by the angle θ between α_j and α_k:*
  - *0 edges if $\theta = \frac{\pi}{2}$,*
  - *1 edge if $\theta = \frac{2\pi}{3}$,*
  - *2 edges if $\theta = \frac{3\pi}{4}$,*
  - *3 edges if $\theta = \frac{5\pi}{6}$.*
- *An arrow from v_j to v_k if α_j and α_k are non-orthogonal and $|\alpha_j| > |\alpha_k|$.(In this note, we also use black points to stand for the smaller one and white one to stand for larger one.)*

Two Dynkin diagrams are said to be **isomorphic** if there is a one-to-one, onto map of the vertices of one to the vertices of the other that preserves the number of bonds and the direction of the arrows.

By Proposition 1.5.7, we know that for any two bases Δ_1 and Δ_2 for R, there exits a unique $w \in W$ shuch that $w\Delta_1 = \Delta_2$. Since w preserves angles and lengths, the Dynkin diagrams associated to two different bases for the same root system are isomorphic.

Proposition 1.6.2.
1. *A root system R is irreducible if and only if its Dynkin diagram is connected.*
2. *If the Dynkin diagrams of two root systems R_1 and R_2 are isomorphic, then R_1 and R_2 are isomorphic.*

Proof. **Part 1:**

($\Rightarrow$) If R decomposes as $R = R_1 \oplus R_2$, choose bases $\Delta = \Delta_1 \cup \Delta_2$. Since roots in R_1 are orthogonal to R_2, the Dynkin diagram of Δ is disconnected.

($\Leftarrow$) If the Dynkin diagram is disconnected, decompose $\Delta = \Delta_1 \sqcup \Delta_2$. Let $E_1 = \mathrm{Span}(\Delta_1)$, $E_2 = \mathrm{Span}(\Delta_2)$. Define $R_i = R \cap E_i$. Then $R = R_1 \oplus R_2$, and $W = W_1 \times W_2$. Since $W \cdot \Delta = R$, all roots lie in E_1 or E_2, proving reducibility.

Part 2: Assume R_1, R_2 irreducible with connected Dynkin diagrams. Let $\Delta_1 = \{\alpha_1, \ldots, \alpha_r\}$, $\Delta_2 = \{\beta_1, \ldots, \beta_r\}$. Rescale inner products so $\|\alpha_1\| = \|\beta_1\|$. The isomorphism of diagrams implies $\langle \alpha_j, \alpha_k \rangle = \langle \beta_j, \beta_k \rangle$. Define a linear isometry $A : E_1 \to E_2$ by $A\alpha_j = \beta_j$. For any $\alpha \in R_1$:

$$\alpha = s_{\alpha_{j_1}} \cdots s_{\alpha_{j_N}} \alpha_k \implies A\alpha = s_{\beta_{j_1}} \cdots s_{\beta_{j_N}} \beta_k \in R_2.$$

Similarly $A^{-1}(R_2) \subseteq R_1$. Thus A is a root system isomorphism. □

To study the properties of Dykin diagram, we use a matrix to represent the number of lines between roots in the Dykin diagram.

Definition 1.6.3. *Let (E, R) be a root system with a baes $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. The **Cartan matrix** $A = (a_{ij})$ is an $n \times n$ integer matrix defined by:*

$$a_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}, \quad 1 \leq i, j \leq n,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on the root system.

By Proposition 1.1.6 (and its proof), we find that if α_i and α_j are not multiples, $a_{ij}a_{ji}$ is the number of edges between α_i and α_j.

Also, we can have such properties of Cartan matrix:

1. $a_{ii} = 2$.
2. $a_{ij} \leq 0$ for any $i \neq j$.
3. $det(A) \neq 0$

Chapter 2

Classification of simple Lie algebras

2.1 Basic definitions and theorems about simple Lie algebras

Definition 2.1.1. *A **finite-dimensional real or complex Lie algebra** is a finite-dimensional real or complex vector space $\mathfrak{g}$, together with a map $[\cdot,\cdot]$ from $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$, with the following properties:*

1. *$[\cdot,\cdot]$ is bilinear.*
2. *$[\cdot,\cdot]$ is skew symmetric: $[X,Y] = -[Y,X]$ for all $X, Y \in \mathfrak{g}$.*
3. *The **Jacobi identity** holds:*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$.

*Two elements X and Y of a Lie algebra $\mathfrak{g}$ **commute** if $[X,Y] = 0$. A Lie algebra $\mathfrak{g}$ is **commutative** if $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$.*

Definition 2.1.2. *A Lie algebra $\mathfrak{g}$ is called **irreducible** if the only ideals in $\mathfrak{g}$ are $\mathfrak{g}$ and $\{0\}$. A Lie algebra $\mathfrak{g}$ is called **simple** if it is irreducible and $\dim \mathfrak{g} \geq 2$.*

For the definitions and theorems below, most of them holds for semi-simple Lie algebra too. However, in this report, we only focus on the simple Lie algebra.

Definition 2.1.3. *If $\mathfrak{g}$ is a complex simple Lie algebra, then a **Cartan subalgebra** of $\mathfrak{g}$ is a complex subspace of g with the following properties:*

1. *For all $H_1, H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$.*
2. *If, for some $X \in \mathfrak{g}$, we have $[H,X] = 0$ for all $H \in \mathfrak{h}$, then X is in $\mathfrak{h}$.*
3. *For all H in $\mathfrak{h}$, ad_H is diagonalizable.*

According to our discussion in class, we will give such propositions without proving:

Proposition 2.1.4. *Let $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ be a complex simple Lie algebra and let $\mathfrak{t}$ be any maximal commutative subalgebra of $\mathfrak{k}$. Define $\mathfrak{h} \subset \mathfrak{g}$ by*

$$\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} = \mathfrak{t} + i\mathfrak{t}.$$

Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

Proposition 2.1.5. *We can define the vector space $E = i\mathfrak{t}$ and the a nonzero element α of $\mathfrak{h}$ is a **root** (for $\mathfrak{g}$ relative to $\mathfrak{h}$) if there exists a nonzero $X \in \mathfrak{g}$ such that*

$$[H, X] = \langle \alpha, H \rangle X$$

for all $H \in \mathfrak{h}$.The set of all roots is denoted R.

Then, such (E,R) is a well-defined root system.

2.2 The Root Systems of the Classical Lie Algebras and their Dykin diagrams

2.2.1 The Special Linear Algebras $\mathfrak{sl}(n+1, \mathbb{C})$

We work with the compact real form $\mathfrak{t} = \mathfrak{su}(n+1)$ and the commutative subalgebra $\mathfrak{t}$ defined as the intersection of diagonal matrices with $\mathfrak{su}(n+1)$:

$$\mathfrak{t} = \left\{ \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_{n+1} \end{pmatrix} \;\middle|\; a_j \in \mathbb{R},\ \sum_{j=1}^{n+1} a_j = 0 \right\}. \tag{7.19}$$

Consider $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$, described as:

$$\mathfrak{h} = \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n+1} \end{pmatrix} \;\middle|\; \lambda_j \in \mathbb{C},\ \sum_{j=1}^{n+1} \lambda_j = 0 \right\}. \tag{7.20}$$

Let E_{jk} denote the matrix with 1 in the jth row and kth column, and 0 elsewhere. For $H \in \mathfrak{h}$ as in (7.20), direct calculation shows:

$$[H, E_{jk}] = (\lambda_j - \lambda_k) E_{jk}. \tag{7.21}$$

For $j \neq k$, $E_{jk} \in \mathfrak{sl}(n+1; \mathbb{C})$ is a simultaneous eigenvector for ad_H with eigenvalue $\lambda_j - \lambda_k$. Every $X \in \mathfrak{sl}(n+1; \mathbb{C})$ decomposes uniquely into diagonal (Cartan subalgebra) and off-diagonal (linear combinations of E_{jk}) parts.

Identifying $\mathfrak{h}$ with the subspace of $\mathbb{C}^{n+1}$ where components sum to zero, the inner product $(X, Y) = \mathrm{trace}(X^* Y)$ restricts to the standard inner product on $\mathbb{C}^{n+1}$. Transferring roots from $\mathfrak{h}^*$ to $\mathfrak{h}$, we obtain:

$$\alpha_{jk} = e_j - e_k \quad (j \neq k),$$

where e_j are standard basis vectors. The roots of $\mathfrak{sl}(n+1;\mathbb{C})$ form the A_n root system. Each root has length $\sqrt{2}$, and the inner product $(\alpha_{jk}, \alpha_{j'k'})$ takes values in $\{0, \pm 1, \pm 2\}$, depending on the overlap of $\{j,k\}$ and $\{j',k'\}$. The angle between non-proportional roots α, β is $\pi/3$, $\pi/2$, or $2\pi/3$.

Now, we can compute the Cartan matrix of such root system A_n:

$$\begin{pmatrix} 2 & -1 & 0 & & & . \\ -1 & 2 & -1 & & & . \\ 0 & -1 & . & & & . \\ & & & . & -1 & 0 \\ & & & -1 & 2 & -1 \\ . & . & . & 0 & -1 & 2 \end{pmatrix}$$

.

Then, we get the Diagram of the root system A_n:

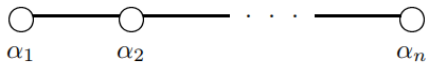


Figure 2.1: the Diagram of the root system A_n

2.2.2 The Orthogonal Algebras $\mathfrak{so}(2n;\mathbb{C})$

The root system for $\mathfrak{so}(2n;\mathbb{C})$ is denoted D_n. Consider the compact real form $\mathfrak{so}(2n)$ and its commutative subalgebra $\mathfrak{t}$ consisting of 2×2 block-diagonal matrices where the jth diagonal block is:

$$\begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix}, \quad a_j \in \mathbb{R}. \tag{7.22}$$

Define $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ in $\mathfrak{so}(2n;\mathbb{C})$, consisting of similar blocks with $a_j \in \mathbb{C}$. $\mathfrak{t}$ is maximal commutative in $\mathfrak{so}(2n)$, making $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{so}(2n;\mathbb{C})$.

The root vectors are 2×2 block matrices with a matrix C in the (j,k)-block ($j<k$), $-C^\top$ in the (k,j)-block, and zeros elsewhere. Here C is one of:

$$C_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

Direct calculation confirms these are root vectors, with corresponding roots:

$$i(a_j + a_k),\ -i(a_j + a_k),\ i(a_j - a_k),\ -i(a_j - a_k).$$

Using the inner product $(X,Y) = \frac{1}{2}\mathrm{trace}(X^*Y)$ on $\mathfrak{h}$, identify $\mathfrak{h}$ with $\mathbb{C}^n$ via:

$$H \mapsto i(a_1, \ldots, a_n).$$

Under this identification, the roots correspond to vectors in $\mathbb{C}^n$:

$$\pm e_j \pm e_k \quad (j < k),$$

where $\{e_j\}$ is the standard basis for $\mathbb{R}^n$.

Now, we can compute the Cartan matrix of such root system D_n:

$$\begin{pmatrix} 2 & -1 & 0 & & & & . \\ -1 & 2 & -1 & & & & . \\ 0 & -1 & . & & & & . \\ & & & . & -1 & 0 & 0 \\ & & & -1 & 2 & -1 & -1 \\ & & & 0 & -1 & 2 & 0 \\ . & . & . & 0 & -1 & 0 & 2 \end{pmatrix}$$

Then, we get the Diagram of the root system D_n:

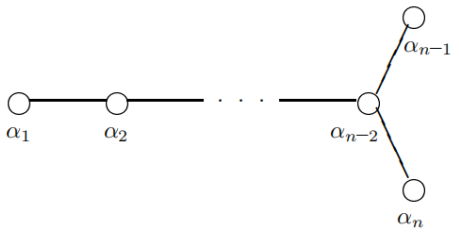


Figure 2.2: the Diagram of the root system D_n

2.2.3 The Orthogonal Algebras $\mathfrak{so}(2n+1;\mathbb{C})$

The root system for $\mathfrak{so}(2n+1;\mathbb{C})$ is denoted B_n. Consider the compact real form $\mathfrak{so}(2n+1)$ and its commutative subalgebra $\mathfrak{t}$ consisting of block-diagonal matrices with n blocks of size 2×2 and one 1×1 block. The 2×2 blocks follow the same structure as in $\mathfrak{so}(2n)$:

$$\begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix}, \quad a_j \in \mathbb{R},$$

while the 1×1 block is zero. Define $\mathfrak{h} := \mathfrak{t}_{\mathbb{C}}$, where the 2×2 blocks allow complex entries. By analogous arguments to the $\mathfrak{so}(2n;\mathbb{C})$ case, $\mathfrak{t}$ is maximal commutative, hence $\mathfrak{h}$ is a Cartan subalgebra.

The Cartan subalgebra of $\mathfrak{so}(2n+1;\mathbb{C})$ is isomorphic to that of $\mathfrak{so}(2n;\mathbb{C})$, both of rank n. All roots of $\mathfrak{so}(2n;\mathbb{C}$ are also roots of $\mathfrak{so}(2n+1;\mathbb{C})$, with $2n$ additional roots. Their root vectors are matrices containing the following blocks:

$$B_1 = \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

placed in positions $(2k, 2n+1)$ and $(2k+1, 2n+1)$, with transposed negatives in $(2n+1, 2k)$ and $(2n+1, 2k+1)$. The corresponding roots in $\mathfrak{h}^*$ are ia_k and $-ia_k$.

Using the inner product $(X, Y) = \frac{1}{2}\text{trace}(X^*Y)$, identify $\mathfrak{h}$ with $\mathbb{C}^n$ as in §7.7.2. The roots of $\mathfrak{so}(2n+1;\mathbb{C})$ are:

$$\pm e_j \pm e_k \ (j < k) \quad \text{and} \quad \pm e_j \ (j = 1, \ldots, n).$$

The roots $\pm e_j$ are shorter by a factor of $\sqrt{2}$ compared to $\pm e_j \pm e_k$.

Now, we can compute the Cartan matrix of such root system B_n:

$$\begin{pmatrix} 2 & -1 & 0 & & & . \\ -1 & 2 & -1 & & & . \\ 0 & -1 & . & & & . \\ & & & . & -1 & 0 \\ & & & -1 & 2 & -2 \\ . & . & . & 0 & -1 & 2 \end{pmatrix}$$

.

Then, we get the Diagram of the root system B_n:

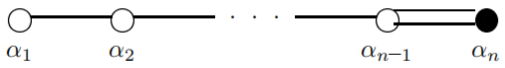


Figure 2.3: the Diagram of the root system B_n

2.2.4 The Symplectic Algebras $\mathfrak{sp}(n;\mathbb{C})$

The root system for $\mathfrak{sp}(n;\mathbb{C})$ is denoted C_n. Define $\mathfrak{sp}(n;\mathbb{C})$ as the space of $2n \times 2n$ complex matrices X satisfying $\Omega X^\top \Omega = X$, where Ω is the $2n \times 2n$ matrix:

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Explicitly, elements of $\mathfrak{sp}(n;\mathbb{C})$ have the form:

$$X = \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix},$$

where A is an arbitrary $n \times n$ matrix, and B, C are symmetric matrices. The compact real form is $\mathfrak{sp}(n) = \mathfrak{sp}(n;\mathbb{C}) \cap \mathfrak{u}(2n)$

The commutative subalgebra $\mathfrak{t} \subset \mathfrak{sp}(n)$ consists of diagonal matrices:

$$\begin{pmatrix} ia_1 & & & & & & \\ & \ddots & & & & & \\ & & ia_n & & & & \\ & & & -ia_1 & & & \\ & & & & \ddots & & \\ & & & & & -ia_n \end{pmatrix}, \quad a_j \in \mathbb{R}.$$

Define $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$, allowing $a_j \in \mathbb{C}$. Calculations confirm $\mathfrak{t}$ is maximal commutative, hence $\mathfrak{h}$ is a Cartan subalgebra.

Root vectors and corresponding roots are as follows:

1. Matrices of the form (for $j \neq k$):

$$\begin{pmatrix} 0 & E_{jk} + E_{kj} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ E_{jk} + E_{kj} & 0 \end{pmatrix} \tag{7.24}$$

   correspond to roots $\pm(a_j + a_k)$.

2. Matrices of the form:

$$\begin{pmatrix} E_{jk} & 0 \\ 0 & -E_{kj} \end{pmatrix} \tag{7.25}$$

   correspond to roots $\pm(a_j - a_k)$.

3. Matrices of the form:

$$\begin{pmatrix} 0 & E_{jj} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ E_{jj} & 0 \end{pmatrix} \tag{7.26}$$

   correspond to roots $\pm 2a_j$.

Using the inner product $(X, Y) = \frac{1}{2}\text{trace}(X^*Y)$, identify $\mathfrak{h}$ with $\mathbb{C}^n$ via:

$$H \mapsto (a_1, \ldots, a_n).$$

The roots become vectors in $\mathbb{C}^n$:

$$\pm e_j \pm e_k \ (j < k) \quad \text{and} \quad \pm 2e_j \ (j = 1, \ldots, n).$$

This root system C_n differs from B_n (for $\mathfrak{so}(2n+1; \mathbb{C})$) by replacing $\pm e_j$ with $\pm 2e_j$.

Now, we can compute the Cartan matrix of such root system C_n:

$$\begin{pmatrix} 2 & -1 & 0 & & & . \\ -1 & 2 & -1 & & & . \\ 0 & -1 & . & & & . \\ & & & . & -1 & 0 \\ & & & -1 & 2 & -1 \\ . & . & . & 0 & -2 & 2 \end{pmatrix}$$

.

Then, we get the Diagram of the root system B_n:

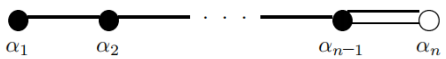


Figure 2.4: the Diagram of the root system B_n

2.3 The Exceptional Lie Algebras

We have displayed the four series of classical Lie algebras and their Dynkin diagrams. And the last question is: Are there any other Lie algebras different from them? The answer is YES! There are only five exceptional Lie algebras, and we are going to prove that.

Theorem 2.3.1. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system R. Then $\mathfrak{g}$ is simple if and only if the Dynkin diagram of R is connected.*

Proof. $\mathfrak{g}$ is simple $\iff$ R is irreducible. By Proposition 1.6.2., R is irreducible $\iff$ its Dynkin diagram is connected. □

Definition 2.3.2. *For a base Δ of (E,R), we say $\alpha_1, \alpha_2 \in \Delta$ are **a joined pair** if they are not orthogonal.*

Lemma 2.3.3. *A base Δ of (E,R) has more points than joined pairs.*

Proof. Consider $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, and let

$$\alpha = \sum(\alpha_i(\langle\alpha_i, \alpha_i\rangle)^{-1/2})$$

Then, $0 < \langle\alpha, \alpha\rangle \implies \sum(a_{ij}a_{ji})^{1/2} <$ number of points in Δ □

By this lemma, we can get an important property of Dykin diagram of a simple Lie algebra.

Corollary 2.3.4. *The Dykin diagram of a simple Lie algebra has no loops.*

Proof. We know that since a Dykin diagram of a simple Lie algebra is connected, it contains a spanning tree, which means that it has at least (n-1) joined pairs. If it has a loop, then it would have n joined pairs, contradiction to Lemma 2.3.3. □

Lemma 2.3.5. *No more than three lines emanate from a vertex of a Dykin diagram of a simple Lie algebra.*

Proof. Consider $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and let $\alpha_2, ..., \alpha_m$ connecte to α_1, with $m > 4$. However, since Δ is a linear independent set, we have:

$$\langle\alpha_1, \alpha_1\rangle > \sum(\langle\alpha_1, \alpha_i\rangle)^2(\langle\alpha_i, \alpha_i\rangle)^{-1}$$

which implies:

$$1 > (1/4)\sum a_{1i}a_{i1}$$

. So, contradiction! □

Corollary 2.3.6. *By Lemma 2.3.5., we can easily find that the only allowable diagram with a triple line is Figure 2.5.*

Theorem 2.3.7. *The Dykin diagram of a simple Lie algebra may have one vertex with three segments meeting at a point, but not more. It may have one double line segment, but not more. It may not have both.*

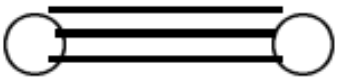


Figure 2.5: the only allowable diagram with a triple line

Proof. In each of these instances, it would be possible to take a subset of the diagram and shrink a chain into a point so that the resulting diagram would have a point with more than three line emanating from it. □

As we mentioned before, for a Cartan matrix A, $det(A) \neq 0$. We will use it to exclude some cases.

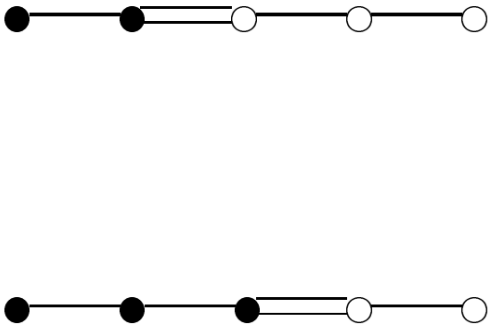


Figure 2.6: Case one

Proposition 2.3.8. *The case in Figure 2.6 is impossible.*

Proof. In this case, we have the Cartan matrix A =

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

, with $det(A) = 0$. Contradiction. □

Corollary 2.3.9. *So, by Proposition 2.3.8., the double line segment is in the middle of the diagram only if $n \leq 4$.*

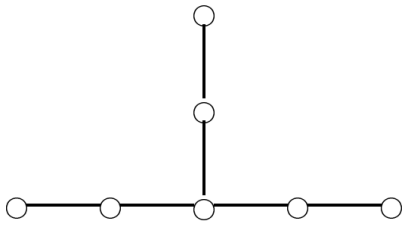


Figure 2.7: Case two

Similarly, we get the folloewing 2 propositions.

Proposition 2.3.10. *The case in Figure 2.7 is impossible.*

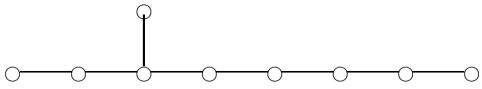


Figure 2.8: Case three

Proposition 2.3.11. *The case in Figure 2.8 is impossible.*

Finally, we can get our final theorem.

Theorem 2.3.12. *There are only nine cases of Dykin diagram of a simple Lie algebra (i.e. in Figure 2.9).*

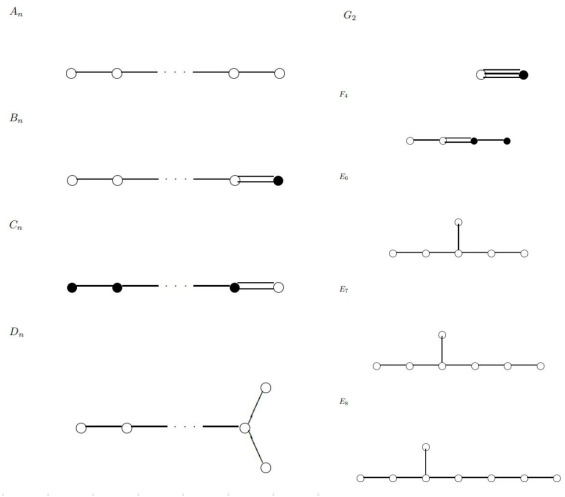


Figure 2.9: nine cases of Dykin diagram of a simple Lie algebra