Representations and BCH formula

Hurui Yao

2025.04.13

Chapter 1

Basic Representation Theory

1.1 Representations

Definition 1.1.1. *Let G be a matrix Lie group. A finite-dimensional complex representation of G is a Lie group homomorphism*

$$\Pi : G \to \mathrm{GL}(V),$$

where V is a finite-dimensional complex vector space (with $\dim(V) \geq 1$). A finite-dimensional real representation of G is a Lie group homomorphism Π of G into $\mathrm{GL}(V)$, where V is a finite-dimensional real vector space.

If $\mathfrak{g}$ is a real or complex Lie algebra, then a finite-dimensional complex representation of $\mathfrak{g}$ is a Lie algebra homomorphism π of $\mathfrak{g}$ into $\mathfrak{gl}(V)$, where V is a finite-dimensional complex vector space. If $\mathfrak{g}$ is a real Lie algebra, then a finite-dimensional real representation of $\mathfrak{g}$ is a Lie algebra homomorphism π of $\mathfrak{g}$ into $\mathfrak{gl}(V)$.

*If Π or π is a one-to-one homomorphism, the representation is called **faithful**.*

Definition 1.1.2. *Let Π be a finite-dimensional real or complex representation of a matrix Lie group G, acting on a space V. A subspace W of V is called **invariant** if $\Pi(A)w \in W$ for all $w \in W$ and all $A \in G$. An invariant subspace W is called **nontrivial** if $W \neq \{0\}$ and $W \neq V$. A representation with no nontrivial invariant subspaces is called **irreducible**.*

The terms invariant, nontrivial, and irreducible are defined analogously for representations of Lie algebras.

Definition 1.1.3. *Let G be a matrix Lie group, let Π be a representation of G acting on the space V, and let Σ be a representation of G acting on the space W. A linear map $\phi : V \to W$ is called an **intertwining map of representations** if*

$$\phi(\Pi(A)v) = \Sigma(A)\phi(v)$$

for all $A \in G$ and all $v \in V$. The analogous property defines intertwining maps of representations of a Lie algebra.

*If ϕ is an intertwining map of representations and, in addition, ϕ is invertible, then ϕ is said to be an **isomorphism of representations**. If there exists an isomorphism between V and W, then the representations are said to be **isomorphic**.*

Proposition 1.1.4. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ and let Π be a representation of G, acting on the space V. Then there is a unique representation π of $\mathfrak{g}$ acting on V such that*

$$\Pi(e^X) = e^{\pi(X)}$$

for all $X \in \mathfrak{g}$. We have

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})$$

and

$$\pi(AXA^{-1}) = \Pi(A)\pi(X)\Pi(A)^{-1}$$

for all $X \in \mathfrak{g}$ and $A \in G$

Proposition 1.1.5. *1. Let G be a connected matrix Lie group with Lie algebra $\mathfrak{g}$. Let Π be a representation of G and π the associated representation of $\mathfrak{g}$. Then Π is irreducible if and only if π is irreducible.*

2. Let G be a connected matrix Lie group, let Π_1 and Π_2 be representations of G, and let π_1 and π_2 be the associated Lie algebra representations. Then π_1 and π_2 are isomorphic if and only if Π_1 and Π_2 are isomorphic.

Proposition 1.1.6. *Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then every finite-dimensional complex representation π of $\mathfrak{g}$ has a unique extension to a complex-linear representation of $\mathfrak{g}_{\mathbb{C}}$, also denoted π. Furthermore, π is irreducible as a representation of $\mathfrak{g}_{\mathbb{C}}$ if and only if it is irreducible as a representation of $\mathfrak{g}$.*

Definition 1.1.7. *If V is a finite-dimensional inner product space and G is a matrix Lie group, a representation $\Pi : G \to \mathrm{GL}(V)$ is **unitary** if $\Pi(A)$ is a unitary operator on V for every $A \in G$.*

Proposition 1.1.8. *Suppose G is a matrix Lie group with Lie algebra $\mathfrak{g}$. Suppose V is a finite-dimensional inner product space, Π is a representation of G acting on V, and π is the associated representation of $\mathfrak{g}$. If Π is unitary, then $\pi(X)$ is skew self-adjoint for all $X \in \mathfrak{g}$. Conversely, if G is connected and $\pi(X)$ is skew self-adjoint for all $X \in \mathfrak{g}$, then Π is unitary.*

1.2 Some Theorems about Representations of Lie groups

1.2.1 Complete Reducibility

Definition 1.2.1. *A finite-dimensional representation of a group or Lie algebra is said to be **completely reducible** if it is isomorphic to a direct sum of a finite number of irreducible representations.*

Definition 1.2.2. *A group or Lie algebra is said to have the **complete reducibility property** if every finite-dimensional representation of it is completely reducible.*

Proposition 1.2.3. *If V is a completely reducible representation of a group or Lie algebra, then the following properties hold.*

1. For every invariant subspace U of V, there is another invariant subspace W such that V is the direct sum of U and W.

2. Every invariant subspace of V is completely reducible.

Theorem 1.2.4. *If G is a compact matrix Lie group, every finite-dimensional representation of G is completely reducible.*

1.2.2 Schur's Lemma

Theorem 1.2.5 ((Schur's Lemma)). *Let G be a Lie group with a Lie algebra $\mathfrak{g}$. Then,*

1. *Let V and W be irreducible real or complex representations of G or $\mathfrak{g}$ and let $\phi : V \to W$ be an intertwining map. Then either $\phi = 0$ or ϕ is an isomorphism.*
2. *Let V be an irreducible complex representation of G or $\mathfrak{g}$ and let $\phi : V \to V$ be an intertwining map of V with itself. Then $\phi = \lambda I$, for some $\lambda \in \mathbb{C}$.*
3. *Let V and W be irreducible complex representations of G or $\mathfrak{g}$ and let $\phi_1, \phi_2 : V \to W$ be nonzero intertwining maps. Then $\phi_1 = \lambda\phi_2$, for some $\lambda \in \mathbb{C}$.*

Corollary 1.2.6. *Let Π be an irreducible complex representation of a matrix Lie group G. If A is in the center of G, then $\Pi(A) = \lambda I$, for some $\lambda \in \mathbb{C}$. Similarly, if π is an irreducible complex representation of a Lie algebra $\mathfrak{g}$ and if X is in the center of $\mathfrak{g}$, then $\pi(X) = \lambda I$.*

Corollary 1.2.7. *An irreducible complex representation of a commutative group or Lie algebra is one dimensional.*

1.3 Representations of sl(2; C) and sl(3; C)

1.3.1 Representation of sl(2; C)

Consider a basis for $\mathbf{sl}(2;\mathbb{C})$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with commutation relations:

$$\begin{cases} [H, X] = 2X, \\ [H, Y] = -2Y, \\ [X, Y] = H. \end{cases} \tag{4.11}$$

Let V be a finite-dimensional complex vector space. If operators A, B, C on V satisfy:

$$[A, B] = 2B, \quad [A, C] = -2C, \quad [B, C] = A,$$

then the unique linear map $\pi : \mathbf{sl}(2;\mathbb{C}) \to \mathbf{gl}(V)$ defined by:

$$\pi(H) = A, \quad \pi(X) = B, \quad \pi(Y) = C,$$

is a representation of $\mathbf{sl}(2;\mathbb{C})$.

Lemma 1.3.1. *Let u be an eigenvector of $\pi(H)$ with eigenvalue $\alpha \in \mathbb{C}$. Then:*

$$\pi(H)\pi(X)u = (\alpha + 2)\pi(X)u,$$

so $\pi(X)u$ is either zero or an eigenvector of $\pi(H)$ with eigenvalue $\alpha + 2$. Similarly:

$$\pi(H)\pi(Y)u = (\alpha - 2)\pi(Y)u,$$

and $\pi(Y)u$ is either zero or an eigenvector with eigenvalue $\alpha - 2$.

Theorem 1.3.2. *For each integer $m \geq 0$, there exists an irreducible complex representation of* $\mathbf{sl}(2;\mathbb{C})$ *with dimension $m+1$. Any two irreducible representations of the same dimension are isomorphic. If π is such a representation, it is isomorphic to π_m defined in Section 4.2.*

Theorem 1.3.3. *Let (π, V) be a finite-dimensional representation of* $\mathbf{sl}(2;\mathbb{C})$. *Then:*

1. *Every eigenvalue of $\pi(H)$ is an integer. If $\pi(X)v = 0$ for an eigenvector v with eigenvalue λ, then λ is non-negative.*
2. *The operators $\pi(X)$ and $\pi(Y)$ are nilpotent.*
3. *The operator $S = e^{\pi(X)}e^{-\pi(Y)}e^{\pi(X)}$ satisfies:*

$$S\pi(H)S^{-1} = -\pi(H).$$

4. *If k is an eigenvalue of $\pi(H)$, then so are $-|k|, -|k|+2, \ldots, |k|-2, |k|$.*

1.3.2 Representation of sl(3; C)

We will use the following basis for $\mathfrak{sl}(3;\mathbb{C})$:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Definition 1.3.4. *If (π, V) is a representation of sl(3; C),then an ordered pair $\mu = (m_1, m_2)$ is called a* ***weight*** *for π if there exists $v \neq 0$ such that*

$$\pi(H_1)v = m_1 v, \pi(H_2)v = m_2 v$$

A non-zero vector v satisfying $\pi(H_i)v = \mu_i v$ for $i = 1, 2$ is called a ***weight vector*** *corresponding to the weight μ. The* ***multiplicity*** *of a weight is the dimension of its corresponding weight space.*

Clearly, we can know that

1. Every representation of sl(3; C) has at least 1 weight.
2. If (π, V) is a representation of sl(3; C) and $\mu = (m_1,m_2)$ is a weight of V, then both m_1 and m_2 are integers.

Definition 1.3.5. *An ordered pair $\alpha = (a_1, a_2) \in \mathbb{Z}^2$ is called a* ***root*** *if:*

1. *a_1 and a_2 are not both zero,*

2. *There exists a non-zero* $Z \in \mathfrak{sl}(3;\mathbb{C})$ *such that*

$$[H_1, Z] = a_1 Z, \quad [H_2, Z] = a_2 Z$$

The element Z *is called a* ***root vector*** *corresponding to the root* α*.*

Lemma 1.3.6. *Let* $\alpha = (a_1, a_2)$ *be a root with root vector* Z_α*, and let* $\mu = (m_1, m_2)$ *be a weight with corresponding weight vector* $v \neq 0$*. Then:*

$$\pi(H_1)(\pi(Z_\alpha)v) = (m_1 + a_1)\pi(Z_\alpha)v, \quad \pi(H_2)(\pi(Z_\alpha)v) = (m_2 + a_2)\pi(Z_\alpha)v$$

Thus, either $\pi(Z_\alpha)v = 0$ *or* $\pi(Z_\alpha)v$ *is a new weight vector with weight*

$$\mu + \alpha = (m_1 + a_1, m_2 + a_2)$$

Definition 1.3.7. *Let* $\alpha_1 = (2, -1)$ *and* $\alpha_2 = (-1, 2)$ *be the simple roots from (6.4). For weights* μ_1 *and* μ_2*, we say* $\mu_1 > \mu_2$ *if* $\mu_1 - \mu_2 = a\alpha_1 + b\alpha_2$ *with* $a \geq 0$ *and* $b \geq 0$*. This relation is denoted:*

$$\mu_1 \succ \mu_2$$

A weight μ *is called a* ***highest weight*** *if it is maximal under this ordering.*

Theorem 1.3.8. *For an irreducible representation* (π, V) *of* $\mathfrak{sl}(3;\mathbb{C})$*:*

1. V *is the direct sum of its weight spaces:* $V = \bigoplus_\mu V_\mu$
2. *There exists a unique highest weight* $\lambda = (\lambda_1, \lambda_2)$
3. *The highest weight vector is unique up to scalar multiples*
4. *All weights satisfy* $\lambda - \mu \in \mathbb{N}\alpha_1 + \mathbb{N}\alpha_2$

Chapter 2

BCH Formula and its Consequences

2.1 The Derivative of the Exponential Map

2.1.1 Adjoint Action on Lie Group and Lie Algebra

Definition 2.1.1. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. The **adjoint representation** (or adjoint action) of G is defined as the map:*

$$\mathrm{Ad}_A : G \to \mathrm{GL}(\mathfrak{g}), \quad \mathrm{Ad}_A(X) = AXA^{-1}$$

for any $A \in G$ and $X \in \mathfrak{g}$. Similarly, the Lie algebra representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is defined by:

$$\mathrm{ad}_Z(Y) = [Z, Y] = ZY - YZ \quad \forall Z, Y \in \mathfrak{g}.$$

Lemma 2.1.2 (Relation between Ad and ad)**.** *For any $X \in \mathfrak{g}$, the following holds:*

$$\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X}.$$

Specifically, the derivative satisfies:

$$\frac{d}{dt}\mathrm{Ad}_{e^{tX}}(Y)\bigg|_{t=0} = \mathrm{ad}_X(Y) \quad \forall Y \in \mathfrak{g}.$$

Expressed as a power series:

$$\mathrm{Ad}_{e^{tX}} = \sum_{k=0}^{\infty} \frac{(t \cdot \mathrm{ad}_X)^k}{k!}.$$

2.1.2 Integral Expression of the Derivative

Lemma 2.1.3 (Derivative of Matrix Exponential)**.** *For any smooth matrix-valued function $A(t)$, the derivative of $e^{A(t)}$ is:*

$$\frac{d}{dt}e^{A(t)} = \int_0^1 e^{(1-s)A(t)} \left(\frac{dA}{dt}\right) e^{sA(t)}\, ds.$$

Proof. Expand $e^{A(t)}$ as a power series:

$$e^{A(t)} = I + A(t) + \frac{A(t)^2}{2!} + \frac{A(t)^3}{3!} + \cdots .$$

Differentiate term-by-term and rearrange using the Beta function identity:

$$\int_0^1 (1-s)^m s^n \, ds = \frac{m!n!}{(m+n+1)!}.$$

This matches the series expansion of the integral expression. □

2.1.3 Derivative of the Exponential Map

Theorem 2.1.4 (Derivative Formula)**.** *For $X, Y \in \mathfrak{g}$, the derivative of e^{X+tY} at $t = 0$ is:*

$$\frac{d}{dt} e^{X+tY}\Big|_{t=0} = e^X \left(\frac{I - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X} \right)(Y) = e^X \left(Y - \frac{[X,Y]}{2!} + \frac{[X,[X,Y]]}{3!} - \cdots \right).$$

More generally, for a smooth path $X(t)$:

$$\frac{d}{dt} e^{X(t)} = e^{X(t)} \left(\frac{I - e^{-\mathrm{ad}_{X(t)}}}{\mathrm{ad}_{X(t)}} \left(\frac{dX}{dt} \right) \right).$$

Proof. Apply Lemma 2 with $A(t) = X + tY$. Using the relation $\mathrm{Ad}_{e^{-sX}} = e^{-s\mathrm{ad}_X}$, the integral becomes:

$$e^X \int_0^1 e^{-s\mathrm{ad}_X}(Y)\, ds = e^X \left(\sum_{k=0}^{\infty} \frac{(-1)^k (\mathrm{ad}_X)^k}{(k+1)!} \right)(Y).$$

This simplifies to the stated series expansion. □

2.2 The Baker-Campbell-Hausdorff Formula

Theorem 2.2.1 (Baker-Campbell-Hausdorff)**.** *For all $n \times n$ complex matrices X and Y with $\|X\|$ and $\|Y\|$ sufficiently small, we have*

$$\log(e^X e^Y) = X + \int_0^1 g\left(e^{\mathrm{ad}_X} e^{t \cdot \mathrm{ad}_Y}\right)(Y)\, dt,$$

where

$$g(z) = \frac{\log z}{1 - z^{-1}} = \sum_{m=0}^{\infty} a_m (z-1)^m.$$

Proof. For sufficiently small X and Y in $M_n(\mathbb{C})$, define

$$Z(t) = \log(e^X e^{tY})$$

for $0 \le t \le 1$. And we use Thm 2.1.4 to compute Z(1). □

2.3 Group Versus Lie Algebra Homomorphisms

Definition 2.3.1 (Local Homomorphism)**.** *Let G and H be matrix Lie groups. A **local homomorphism** from G to H is a pair (U, f), where U is a path-connected neighborhood of the identity in G, and $f : U \to H$ is a continuous map satisfying:*

$$f(AB) = f(A) \cdot f(B)$$

whenever $A, B, AB \in U$.

Proposition 2.3.2 (Construction of Local Homomorphism)**.** *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Define a neighborhood $U_\epsilon \subseteq G$ by:*

$$U_\epsilon = \left\{ A \in G \,\middle|\, \|A - I\| < 1 \text{ and } \|\log A\| < \epsilon \right\}.$$

Then there exists $\epsilon > 0$ such that the map $f : U_\epsilon \to H$, given by:

$$f(A) = e^{\phi(\log A)},$$

is a local homomorphism.

Proof. Choose $\epsilon > 0$ small enough that for any $A \in V_\epsilon$, where $V_\epsilon = e^{U_\epsilon}$ and $U_\epsilon = X \in M_n(\mathbb{C}), ||X|| < \epsilon$, A is in G if and only if $\log A$ is in $\mathfrak{g}$ and for any A, B in U_ϵ, the BCH formula holds for X = $\log A$, Y = $\log B$.

Then, if AB $\in U_\epsilon$,

$$f(AB) = f(e^x e^y) = e^{\phi(\log(e^x e^y))}.$$

$$\log(e^x e^y) = X + \int_0^4 g(e^{\mathrm{ad}_x} e^{t^a\, dx})(Y)\, dt$$

$$\Rightarrow \phi(\log(e^x e^y)) = \phi(X) + \int_0^4 g(e^{\mathrm{ad}_{\phi(x)}} e^{t^a\, d_{\phi(y)}})(\phi(Y))\, dt = \log(e^{\phi(x)} e^{\phi(y)}), \text{ since } \phi \text{ is a}$$

Lie algebra homomorphism, ad *is a representation of* $\mathfrak{g}$ and $g(z) = \sum_{m=0}^{\infty} a_m (z-1)^m$.

So, $f(AB) = e^{\log(e^{\phi(X)} e^{\phi(Y)})} = e^{\phi(X)} e^{\phi(Y)} = f(A) f(B)$ □

Theorem 2.3.3 (Global Extension of Local Homomorphism)**.** *Let G and H be matrix Lie groups with G simply connected. If (U, f) is a local homomorphism from G to H, there exists a unique global Lie group homomorphism $\Phi : G \to H$ that agrees with f on U.*

Proof. We prove this thm by 4 steps:

1. Define $\Phi : G \to H$ along a path.
2. Prove the $\Phi : G \to H$ is independent of partition
3. Prove the $\Phi : G \to H$ is independent of the path with the same end points
4. Prove the $\Phi : G \to H$ is a homomorphism and agrees with f on U.

□

Theorem 2.3.4 (Lifting Lie Algebra Homomorphisms)**.** *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, there exists a unique Lie group homomorphism $\Phi : G \to H$ such that:*

$$\Phi(e^X) = e^{\phi(X)} \quad \forall X \in \mathfrak{g}.$$

Corollary 2.3.5 (Isomorphism of Simply Connected Groups)**.** *Suppose G and H are simply connected matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. If $\mathfrak{g}$ is isomorphic to $\mathfrak{h}$, then G is isomorphic to H.*

Theorem 2.3.6 (Decomposition of Simply Connected Groups)**.** *Let G be a simply connected matrix Lie group with Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{g}$ decomposes as a Lie algebra direct sum $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are subalgebras. Then there exist closed, simply connected subgroups H_1 and H_2 of G with Lie algebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. Moreover, G is isomorphic to the direct product $H_1 \times H_2$.*